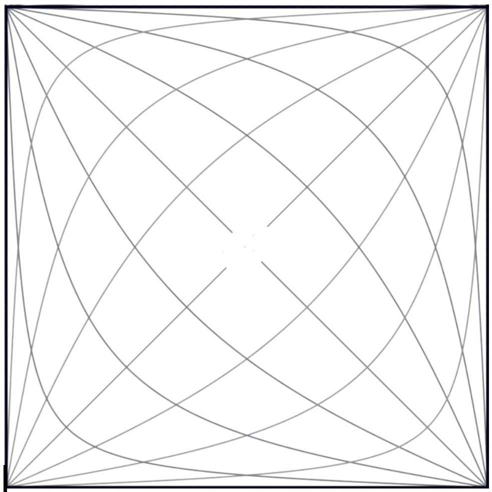
A Guide to

Tetrads in General Relativity



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Abstract

This guide is a review of the Einstein vierbein formalism and its generalization to the Einstein-Cartan theory. Central is Einstein's equivalence principle that special relativity is valid locally in a freely falling observer frame. The understanding of this principle as a gauge symmetry leads to general relativity theory (GRT) with the Riemann curvature tensor the corresponding gauge-field strength. More generally, gauging the Poincaré-group symmetry of special relativity, Kibble and Sciama derived an extension of GRT, called the Einstein-Cartan theory, embedded in a Riemann-Cartan spacetime with torsion. The mathematical language in this review is Geometric Algebra (GA). It structures tetrad frames spanning the spacetime tangent spaces by the Clifford algebra $\mathrm{Cl}(1,3)$, including the algebra of differential forms as a sub-algebra.

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Preface

Introduction

> *The essence of General Relativity not necessarily consists in the application of Riemannian geometry to physical space and time. In my view, it is much more useful to regard General Relativity above all as a theory of gravitation, whose connection with geometry arises from the peculiar empirical properties of gravitation, properties summarized by Einstein's Principle of the Equivalence of Gravitation and Inertia.* Steven Weinberg, 1972

The conventional formulation of General Relativity is based on a Riemannian geometry of a torsion-free spacetime with a covariantly conserved metric. The *vierbein* formalism, introduced by Albert Einstein in 1928 and independently developed by Hermann Weyl in 1929, generalizes the choice of basis for the tangent space from a coordinate basis to the less restrictive choice of a local basis, i.e. a locally defined set of four linearly independent vector fields known as *tetrads.* This geometric structure exists independently of any prior gravity-model assumptions, facilitating a local description through inertial frames at each point.

Physically, these local inertial frames represent the standard laboratory setups employed by observers conducting measurements in space and time. From a given local frame, different classes of observer frames may be obtained by performing local (point-dependent) Lorentz transformations. These include transformations that boost away local gravity. The statement that these different tetrad bases are equivalent, may be viewed as the *gauge principle* of local Lorentz symmetry. It encompasses the essence of Einstein's equivalence principle that gravity may be annulled by acceleration.

In the further development of general relativity, the mathematical insights of Élie Cartan (1869-1951) had a major influence; in particular, his theory of *differential forms* (1899) and the generalized concept of the *Maurer-Cartan connection* (1922-1924) with its corresponding *curvature* and *torsion.* By allowing *torsion*, Einstein's vierbein formulation becomes a Poincaré gauge theory of gravitation in a *Riemann-Cartan geometry.*

In 1913 Cartan introduced *spinors* in geometry before their physical relevance was discovered by Paul Dirac. In the context of General Relativity, and its extension to the Einstein-Cartan Theory, the tetrad formalism proves especially advantageous when working with spinors. It offers a natural and coherent framework for describing spinor fields and their interactions with gravity, seamlessly linking with the Dirac equation in curved spacetime.

Mathematically the tetrad formalism is augmented by elevating the tangent spaces of the spacetime manifold to a *Geometric Algebra* (GA). This framework naturally accommodates arbitrary bases and encompasses the theory of differential forms. Additionally, calculations using GA are often more straightforward than those in traditional approaches. The GA formalism does not alter the underlying physics; it serves as an effective mathematical tool in this guide.

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Chapter 1

Metric and Connection

1.1 Spacetime [MTW,DH]

☐ In general relativity theory (GRT), spacetime is modeled as an orientable smooth curved Lorentzian (pseudo-Riemannian) 4d manifold $\mathcal{M}_4 = \{\mathbf{x}\}$, parameterized by *coordinates* $\{x^\mu; \mu = 0, 1, 2, 3\}$ so that $\mathbf{x} = \mathbf{x}(x^0, x^1, x^2, x^3) := \mathbf{x}(x)$. The chosen metric signature is $(+, -, -, -)$, one dimension of time and three of space; units such that the velocity of light and the gravitational constant both have the value one: $c = 1$, $G = 1$; where appropriate these constants are written explicitly.

☐ A set of four linearly independent local *coordinate basis vectors* is defined by:

1.1

$$\mathbf{g}_\mu(\mathbf{x}) := \frac{\partial \mathbf{x}}{\partial x^\mu} := \partial_\mu \mathbf{x}$$

At each point $\mathbf{x}$ of the spacetime manifold these *tangent vectors* $\{\mathbf{g}_\mu(\mathbf{x})\}$ provide a *holonomic basis* for the vector space $V_4(\mathbf{x}) = T_\mathbf{x}\mathcal{M}$, called the *tangent space* to $\mathcal{M}_4$ at $\mathbf{x}$. The tangent space is by definition a *Minkowski space*. The *tangent bundle* is the set of tangent spaces at all points of spacetime. Tangent bundle and manifold are said to be *soldered*, because they are both four-dimensional spacetimes. This geometrical structure is intrinsic to the coordinatization of the manifold, independent of any prior assumptions.

☐ Physical quantities of interest in GRT are represented by tensor fields, or other geometric objects like spinors, that live in the tangent vector space attached to each point in spacetime, including the higher-order tensor spaces constructed therefrom. These geometric objects can be specified in terms of arbitrary coordinates, but are themselves coordinate independent. It is a fundamental principle in physics that equations between such geometric objects are *generally covariant*, that is, they are invariant under any change of coordinate system.

☐ In the subject of differential geometry, it is now common practice to identify directional derivatives as tangent vectors. However, there is a complete isomorphism between coordinate vectors $\{\mathbf{g}_\mu\}$ as defined in (1.1) and the corresponding directional derivatives $\{\partial_\mu\}$. For all practical purposes, the two viewpoints are equivalent, and calculations performed in either scheme will return the same results.

1.2 Reciprocal Base [MTW,FK]

☐ The inverse mapping of $\mathbf{x} = \mathbf{x}(x)$ is the set of scalar-valued functions $x^\mu = x^\mu(\mathbf{x})$. The *gradients* of these scalar fields are the vector fields

1.2

$$\mathbf{g}^\mu(\mathbf{x}) := \nabla x^\mu(\mathbf{x})$$

Given a scalar field $f(\mathbf{x})$ defined over some region of the manifold and given a vector $\mathbf{a}$ in the tangent space, the vector-valued gradient $\nabla := \partial_\mathbf{x}$ is defined though the *directional derivative*:

1.3

$$\mathbf{a} \cdot \partial_\mathbf{x} f(\mathbf{x}) := \lim_{\varepsilon \to 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{a}) - f(\mathbf{x})}{\varepsilon}$$

☐ Note that $\mathbf{a} \cdot \partial_\mathbf{x} \mathbf{x} = \mathbf{a}$ when $f(\mathbf{x}) = \mathbf{x}$. Hence:

1.4

$$\mathbf{g}_\mu \cdot \mathbf{g}^\nu = \mathbf{g}_\mu \cdot \nabla x^\nu = \partial x^\nu / \partial x^\mu = \delta^\nu_\mu$$

This implies the gradient relations

1.5

$$\mathbf{g}_\mu \cdot \nabla = \partial_\mu \quad \nabla = \mathbf{g}^\mu \partial_\mu$$

☐ The gradient (cotangent) vectors $\{\mathbf{g}^\mu\}$ as defined in (1.2) form a *reciprocal basis* to the $\{\mathbf{g}_\mu\}$, i.e., the reciprocal basis vectors can be written as a linear sum over the holonomic basis vectors, and vice versa. Both vector bases cover the *same* tangent space and are both referred to as the *natural (holonomic) coordinate basis* at point $\mathbf{x}$, or equivalently, $x = \{x^\mu\}$; there is no need to introduce a dual space to define the inner product. This construction is only possible in *metric spaces*.

1.3 Differential Forms [MTW,FK]

☐ In the case of a metric space, there is an isomorphism between the reciprocal basis $\{\mathbf{g}^\mu\}$ as defined above in (1.2) and the differential forms $\{\mathrm{d}x^\mu\}$, called 1-forms in the *theory of differential forms* pioneered by the mathematician Élie Cartan. For example, in the theory of differential forms the *exterior derivative* of a scalar function (zero-form) $f(x)$ is defined as the gradient 1-form

1.6

$$\mathrm{d}f(x) := (\partial_\mu f)_x \mathrm{d}x^\mu = (\partial_\mu f)_x \mathbf{g}^\mu$$

In particular, it follows that $\mathrm{d}x^\nu = \partial_\mu x^\nu \mathrm{d}x^\mu = \mathrm{d}x^\nu$. Thus, it is seen that the action of the exterior differential on a zero-form is the same as the action of the gradient $\nabla = \mathbf{g}^\mu \partial_\mu$.

□ A general 1-form is a linear, real-valued function $\mathbf{A} = A_\mu \mathrm{d}x^\mu$ with respect to the basis differentials $\mathrm{d}x^\mu$. The exterior derivative d acts as the *curl* $\nabla\wedge$ on the 1-form $\mathbf{A}$ raising the degree to give the 2-form

1.7

$$\mathrm{d}\mathbf{A} := \partial_\mu A_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \partial_\mu A_\nu \mathbf{g}^\mu \wedge \mathbf{g}^\nu$$

This defines an anti-symmetric covariant tensor $F_{\mu\nu} = F_{[\mu\nu]}$ with the differential 2-form

1.8

$$\mathbf{F} := \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \partial_{[\mu} A_{\nu]} \mathbf{g}^\mu \wedge \mathbf{g}^\nu$$

The bracket, like the wedge product, indicates *anti-symmetrization* of the indices with an implicit factor 1/2.

□ Differential forms are in one-to-one correspondence with totally *antisymmetric* tensor fields. These objects are particularly suited for analyzing fields on manifolds and vector spaces in GRT, providing a more flexible and powerful framework than traditional tensor calculus. For example, compared with the tensor formalism, differential forms together with Cartan's structure equations (see First Cartan Equation and Second Cartan Equation) offer an efficient way of obtaining the connection elements. In Chapter III the theory of forms is extended to become a Geometric Algebra (GA), also known as a Clifford Algebra, or SpaceTime Algebra (STA).

1.4 Metric Tensor [MTW,FK]

□ Spacetime is locally *Lorentz invariant*, i.e. the vector space $V_4(x)$, $x = \{x^\mu\}$, is a Minkowski space. Thus, an infinitesimal displacement gives the line element:

1.9

$$ds^2 = d\mathbf{x} \cdot d\mathbf{x} = \mathbf{g}_\mu \cdot \mathbf{g}_\nu dx^\mu dx^\nu := g_{\mu\nu} dx^\mu dx^\nu$$

Here, the components of the *metric tensor field* are defined as as the *inner product* of the coordinate tangent vectors:

1.10

$$g_{\mu\nu}(x) := \mathbf{g}_\mu(x) \cdot \mathbf{g}_\nu(x)$$

In GRT the metric is not given apriori, except for special cases in which some features of the metric are known (static gravitational field, presence of symmetries, etc.) or postulated.

In general the metric tensor must be found from Einstein's field equation(s); see Einstein Equation.

☐ The metric tensor is *dimensionless*, *symmetric* in its indices and non-degenerate: $g := \det[g_{\mu\nu}] \neq 0$. It has

1.11

$$g^{\mu\nu}(x) := \mathbf{g}^{\mu}(x) \cdot \mathbf{g}^{\nu}(x)$$

as its *inverse*:

1.12

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu} \quad \mathbf{g}^{\mu} \cdot \mathbf{g}_{\nu} = \delta^{\mu}_{\nu} \quad \mathbf{g}^{\mu} = g^{\mu\nu}\mathbf{g}_{\nu}$$

The covector $\mathbf{g}^{\mu}$ thus implicitly contains a directed measure on the manifold that defines the *inner product* between vectors in the tangent space:

1.13

$$\mathbf{v} \cdot \mathbf{w} = v^{\mu} w^{\nu} \mathbf{g}_{\mu} \cdot \mathbf{g}_{\nu} = g_{\mu\nu} v^{\mu} w^{\nu} = g^{\mu\nu} v_{\mu} w_{\nu}$$

The inner product is *bi-linear* and *symmetric*.

1.5 Coderivative [MTW,DR]

☐ Equation (1.3) defines $\mathbf{a} \cdot \nabla$ as the *directed derivative* along a vector $\mathbf{a}$. When this directional derivative acts on a tangent vector field there is no guarantee that the resulting vector also lies entirely in the *same* tangent space, even if $\mathbf{a}$ does. In order to restrict to quantities intrinsic to the manifold, one defines a new derivative operator D for curved spacetime, the *covariant vector derivative* or *coderivative* for short, which is the covariant generalization of the gradient ∇. In the literature, the coderivative is often denoted as (bold) ∇ and the covariant directional derivative as (bold) $\nabla_{\mathbf{a}}$.

☐ The coderivative is usually introduced with reference to the local coordinate system by defining the *directional coderivative* $D_{\mu} := \mathbf{g}_{\mu} \cdot D$. By definition, its action on a basis vector returns a new vector in the tangent space which must be a linear sum

1.14

$$D_{\mu}\mathbf{g}_{\nu} := \Gamma^{\kappa}_{\mu\nu}\mathbf{g}_{\kappa} \quad D_{\mu}\mathbf{g}^{\nu} = -\Gamma^{\nu}_{\mu\kappa}\mathbf{g}^{\kappa}$$

determined by a set of displacement fields first introduced by Tullio Levi-Civita (1916)

1.15

$$\Gamma^{\kappa}_{\mu\nu}(\mathbf{x}) := (D_{\mu}\mathbf{g}_{\nu}) \cdot \mathbf{g}^{\kappa} := \mathbf{g}^{\kappa} \cdot \Gamma_{\mu}(\mathbf{g}_{\nu})$$

now called the *Levi-Civita connection coefficients* or *Christoffel symbols* (of the second kind). The placement of the indices follows the convention of [Wikipedia: Christoffel Symbols].

☐ The second equation (1.14) follows from the first. Proof: assume the coefficient in the second equation is some other coefficient $\bar{\Gamma}^{\nu}_{\mu\kappa}$. Using the orthogonality of reciprocal basis vectors one then deduces they must be the same:

1.16

$$\begin{aligned} 0 = D_\kappa(\mathbf{g}_\mu \cdot \mathbf{g}^\nu) &= (D_\kappa \mathbf{g}_\mu) \cdot \mathbf{g}^\nu + \mathbf{g}_\mu \cdot (D_\kappa \mathbf{g}^\nu) \\ &= \Gamma^{\lambda}_{\kappa\mu} \mathbf{g}_\lambda \cdot \mathbf{g}^\nu - \bar{\Gamma}^{\nu}_{\kappa\lambda} \mathbf{g}^\lambda \cdot \mathbf{g}_\mu = \Gamma^{\nu}_{\kappa\mu} - \bar{\Gamma}^{\nu}_{\kappa\mu} \end{aligned}$$

☐ The connection coefficients (1.15) give the rate of change of the basis vector $\mathbf{g}_\nu$ in the direction of the basis vector $\mathbf{g}_\mu$ and may be seen as components of a *linear (affine) map* between tangent spaces. The introduction of this affine structure of the spacetime manifold is mandatory in order to be able to compare tensor quantities at different spacetime points and to define coderivatives. Because of the principle of covariance, differential operators in physical laws describing the evolution of fields have to be covariant.

1.6 Levi-Civita Connection [SW,DR]

☐ On *scalar functions* the coderivative $D_\mu = \mathbf{g}_\mu \cdot D, \quad D = \mathbf{g}^\mu D_\mu,$ acts as an ordinary spacetime derivative $D_\mu f(\mathbf{x}) := \partial_\mu f(\mathbf{x})$. With this rule and the Levi-Civita connection as defined in (1.14), covariant derivatives of a vector $\mathbf{A}$ are calculated as:

1.17

$$D_\mu \mathbf{A} = D_\mu(A^\nu \mathbf{g}_\nu) = (\partial_\mu A^\nu + \Gamma^{\nu}_{\mu\kappa} A^\kappa)\mathbf{g}_\nu := A^\nu{}_{;\mu}\mathbf{g}_\nu$$

1.18

$$D \cdot \mathbf{A} = \mathbf{g}^\mu D_\mu \cdot (A^\nu \mathbf{g}_\nu) = \partial_\mu A^\mu + \Gamma^{\mu}_{\mu\kappa} A^\kappa$$

1.19

$$\mathbf{a} \cdot D\mathbf{A} = a^\mu D_\mu(A^\nu \mathbf{g}_\nu) = a^\mu(\partial_\mu A^\nu + \Gamma^{\nu}_{\mu\kappa} A^\kappa)\mathbf{g}_\nu$$

1.20

$$D \wedge \mathbf{A} = \mathbf{g}^\mu \wedge D_\mu(A_\nu \mathbf{g}^\nu) = \partial_\mu A_\nu \mathbf{g}^\mu \wedge \mathbf{g}^\nu + A_\nu D \wedge \mathbf{g}^\nu$$

☐ The last term of (1.20) is *zero* in GRT; see equation (1.26). If and only if that is the case, the covariant curl $D\wedge$ is equivalent to the exterior derivative d in (1.7). Note the standard notation ';' in (1.17) for the covariant derivative in the component tensor formalism.

☐ If $\mathbf{F}$ is the 2-form (1.8), then:

1.21

$$\begin{aligned} D \cdot \mathbf{F} &= \mathbf{g}^\mu \cdot \left[D_\mu \left(\frac{1}{2} F^{\kappa\lambda} \mathbf{g}_\kappa \wedge \mathbf{g}_\lambda \right) \right] \\ &= \left(\partial_\kappa F^{\kappa\lambda} + \Gamma^{\nu}_{\nu\kappa} F^{\kappa\lambda} \right) \mathbf{g}_\lambda := F^{\kappa\lambda}{}_{;\kappa} \mathbf{g}_\lambda \end{aligned}$$

1.22

$$D \cdot D \cdot \mathbf{F} = g^{\mu} \cdot D_{\mu}(F^{\kappa\lambda}{}_{;\kappa}\mathbf{g}_{\lambda}) = F^{\mu\nu}\partial_{\nu}\Gamma^{\kappa}_{\kappa\mu}$$

The last term of (1.22) actually vanishes because $F^{\mu\nu}$ is antisymmetric in its indices, whereas the second factor turns out to be symmetric; see (1.29).

1.7 Torsion Tensor [MTW,DR]

☐ The Levi-Civita connection coefficients Γ depend on the chosen coordinates and are not tensors. However, the difference of two connections can be shown to be tensorial. Hence, the *torsion* defined as the difference

1.23

$$T^{\kappa}_{\mu\nu} := \Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu} = 2\Gamma^{\kappa}_{[\mu\nu]}$$

transforms as a tensor. Equivalently, the torsion 2-form

1.24

$$\mathbf{T}^{\kappa} := \frac{1}{2}T^{\kappa}_{\mu\nu}\mathbf{g}^{\mu} \wedge \mathbf{g}^{\nu} = \Gamma^{\kappa}_{[\mu\nu]}\mathbf{g}^{\mu} \wedge \mathbf{g}^{\nu}$$

measuring the anti-symmetry of the Levi-Civita connection, transforms as a vector. Both representations show that torsion vanishes if and only if the connection coefficients are *symmetric* with respect to the lower indices.

☐ Since the embedding space is smooth, partial spacetime derivatives commute: $0 = \partial_{\mu}\partial_{\nu}\mathbf{x} - \partial_{\nu}\partial_{\mu}\mathbf{x} = \partial_{\mu}\mathbf{g}_{\nu} - \partial_{\nu}\mathbf{g}_{\mu}$. Then, this only holds for the coderivative

1.25

$$D_{\mu}\mathbf{g}_{\nu} - D_{\nu}\mathbf{g}_{\mu} = \left(\Gamma^{\kappa}_{\mu\nu} - \Gamma^{\kappa}_{\nu\mu}\right)\mathbf{g}_{\kappa} = 0$$

if $\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu}$ is *symmetric*. This imposes that the Levi-Civita connection is *torsion-free*, which implies

1.26

$$D \wedge \mathbf{g}^{\mu} = \mathbf{g}^{\nu} \wedge D_{\nu}\mathbf{g}^{\mu} = -\Gamma^{\mu}_{[\nu\kappa]}\mathbf{g}^{\nu} \wedge \mathbf{g}^{\kappa} = 0$$

☐ Torsion vanishes *by assumption* in GRT. Other gravity theories such as, in particular, the Einstein-Cartan Theory, allow for torsion to incorporate possible new effects beyond Einstein gravity.

1.8 Metric Compatibility [SW,MTW]

☐ To establish a relation between the connection coefficients and the metric, one may apply the coderivative to both sides of eq. (1.10). Since the metric is a scalar function

1.27

$$\begin{aligned} D_\kappa g_{\mu\nu} = \partial_\kappa g_{\mu\nu} &= (D_\kappa \mathbf{g}_\mu)\cdot \mathbf{g}_\nu + \mathbf{g}_\mu \cdot (D_\kappa \mathbf{g}_\nu) \\ &= \Gamma^\lambda_{\kappa\mu}\mathbf{g}_\lambda \cdot \mathbf{g}_\nu + \Gamma^\lambda_{\kappa\nu}\mathbf{g}_\lambda \cdot \mathbf{g}_\mu = \Gamma^\lambda_{\kappa\mu} g_{\lambda\nu} + \Gamma^\lambda_{\kappa\nu} g_{\lambda\mu} \end{aligned}$$

It is then seen that the coderivative of the metric vanishes:

1.28

$$g_{\mu\nu;\kappa} := \partial_\kappa g_{\mu\nu} - \Gamma^\lambda_{\kappa\mu} g_{\lambda\nu} - \Gamma^\lambda_{\kappa\nu} g_{\lambda\mu} = 0$$

It follows that the operations of covariant differentiation and raising and lowering of indices *commute*. This result is called the '*metric compatibility*' of the coderivative, i.e., the connection preserves the scalar inner product induced by the metric $g_{\mu\nu}$.

☐ Add to equation (1.28) the same equation with κ and μ interchanged and subtract the same equation with κ and μ interchanged. One then obtains the *Christoffel formulae*:

1.29

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\kappa\mu} - \partial_\kappa g_{\mu\nu}) \qquad \Gamma^\mu_{\mu\nu} = \frac{1}{2} g^{\mu\kappa}\partial_\nu g_{\mu\kappa}$$

If the metric is postulated or can be determined on physical grounds, then a knowledge of the embedding is not necessary for calculating the connection coefficients.

☐ Given a metric connection on a Lorentzian (pseudo-Riemannian) manifold with torsion, one can always find a single connection that is torsion free. This is the *Levi-Civita connection*, uniquely defined as *torsion-free* and *metric-compatible*. The Levi-Civita connection has as many independent components as $g_{\mu\nu;\kappa}$ and is therefore fully determined by the metric compatibility equation (1.28).

Chapter 2

Tetrad and Vierbein

2.1 Tetrad Field [JY]

☐ A *tetrad* (Greek foursome) is a set of axes $\{\mathbf{e}_a(x); a = 0, 1, 2, 3\}$ at a point $x = \{x^\mu\}$ of the spacetime $\mathcal{M}_4$ that span the tangent space $V_4(x) = T_x\mathcal{M}_4$. A common choice is an *orthonormal tetrad*, where the axes form a local frame at each point, so that the scalar products of the axes constitute the Minkowskian metric $\eta_{ab} = (1, -1, -1, -1)$:

2.1

$$\mathbf{e}_a(x) \cdot \mathbf{e}_b(x) = \eta_{ab}$$

By using the matrix inverse of the metric, a reciprocal *orthonormal* tetrad basis may be defined by

2.2

$$\mathbf{e}^a(x) \cdot \mathbf{e}_b(x) = \delta^a_b$$

This construction exists at every point of $\mathcal{M}_4$, independent of the coordinate basis $\{\mathbf{g}_\mu, \mathbf{g}^\nu\}$.

☐ The tetrad base $\mathbf{e}_a(x)$ at each point of spacetime defines a local Lorentz frame with a Minkowski metric. It may be imagined to be the *proper rest frame* of a local (idealized) observer able to measure physical quantities, space and time distances in particular. On assumption of the local nature of physics, the tetrad formalism thus offers an interpretation of GRT in terms of concepts of special relativity; see section Postulates.

☐ The inner product (2.1) is invariant under *local Lorentz transformations* (LLT's)

2.3

$$\mathbf{e}_a(x) \to \mathbf{e}'_a(x) = \Lambda_a{}^b(x)\mathbf{e}_b(x)$$

The matrices $\Lambda_a{}^b(x)$ represent position-dependent inverse Lorentz transformations which operate on the basis vectors. Thus, the tetrad indices $a, b, c, ...$ may be understood as forming the vector representation of the Lorentz group $\mathrm{SO}(1,3)$ with six independent degrees of freedom, three degrees of freedom in spatial rotations, and three more in Lorentz boosts. Tetrad transformations rotate the tetrad axes at each point, while leaving the back-ground coordinates $\{x^\mu\}$ unchanged. So, in the context of GRT the Lorentz group is the *symmetry group* of local tetrad rotations and boosts; see section Gauge Principle.

☐ Tetrads are geometric objects defined independently of coordinates. Tetrad components of tensors therefore do not change when a coordinate transformation is applied. Quantities that are unchanged by a coordinate transformation are *coordinate gauge invariant*. Quantities that are unchanged under a tetrad transformation are *tetrad gauge invariant*. For example, tetrad tensors are coordinate gauge-invariant, while coordinate tensors are tetrad gauge-invariant.

2.2 Vierbein Field [JY,SW]

☐ Associated with the tetrad frame at each point is a set of *local coordinates* x^a, $a \in \{0,1,2,3\}$. Unlike the coordinates x^μ of the background geometry, the local coordinates x^a do not extend beyond the local frame at each point. In these coordinates the scalar spacetime distance is $ds^2 = \eta_{ab} dx^a dx^b$. Writing $dx^a = e_\mu{}^a dx^\mu$ one derives the relation:

2.4

$$g_{\mu\nu}(x) = e_\mu{}^a(x) e_\nu{}^b(x) \eta_{ab}$$

expressing $g_{\mu\nu}(x)$ in terms of Einstein's *vierbein fields* $e_\mu{}^a(x)$ and the flat metric. This way, the spacetime metric may be seen as a deformation of the Minkowskian (tangent space) metric, the product of vierbein fields carrying the metric information. The first-order vierbein $e_\mu{}^a \simeq \partial_\mu x^a$ is called *trivial*. Except for some special cases, e.g. a freely falling system, vierbeins in curved space are non-trivial: $\partial_{[\mu} e_{\nu]}{}^a \neq 0$; see section Structure Coefficients

☐ Being a symmetric rank-2 tensor in $d=4$ dimensions, the metric tensor at left-hand side of (2.4) has $d(d+1)/2$ independent components. On the other hand, since they have no particular symmetries, the number of independent components of the vierbeins is d^2. This means that the choice of the vierbeins is *not unique*. The difference in independent components is $d(d-1)/2$, which matches precisely with the number of generators of the local Lorentz group in d-dimensions. Therefore, all the equivalent choices of the vierbein are related by local Lorentz transformations (2.3).

☐ If the local coordinates x^a are kept fixed at each physical point x, the vierbeins $e_\mu{}^a$ change under *manifold diffeomorphism*, i.e. an invertible and differential map $x^\mu \to x'^\mu$, according to the rule

2.5

$$e_\mu{}^a(x) \to e'_\mu{}^a(x') = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu{}^a(x)$$

Thus, the vierbein field ${e_\mu}^a(x)$ may be thought of as forming four *covariant tetrad fields* $\mathbf{e}^a := {e_\mu}^a \mathbf{g}^\mu$, one for each value of the upper index, with ${e_\mu}^a(x)$ the *components* with respect to the coordinate basis.

2.3 Vierbein and Metric

□ The primitive object in the tetrad formalism is the *vierbein field* ${e_\mu}^a(x)$, in place of the spacetime metric in the coordinate formalism. The vierbeins are real 4×4 invertible matrices, with 16 independent components, acting as a *soldering agent* between the general manifold (Greek indices) and the Minkowski spacetime (Latin indices):

2.6

$$\mathbf{g}_\mu(x) = {e_\mu}^a(x)\mathbf{e}_a(x) \quad {e_\mu}^a(x) := \mathbf{g}_\mu(x) \cdot \mathbf{e}^a(x)$$

2.7

$$\mathbf{e}_a(x) = {e^\mu}_a(x)\mathbf{g}_\mu(x) \quad {e^\mu}_a(x) := \mathbf{g}^\mu(x) \cdot \mathbf{e}_a(x)$$

□ From their definition it follows that the local Lorentz frame vierbein fields ${e_\mu}^a(x)$ diagonalize the metric tensor as in (2.4). In terms of the inverse matrices ${e^\mu}_a(x)$, equation (2.1) becomes the so-called the *inner product-signature constraint*

2.8

$$g_{\mu\nu}(x){e^\mu}_a(x){e^\nu}_b(x) = \eta_{ab} \qquad \sqrt{-g}\,\det[{e^\mu}_a] = 1$$

This key relation is basic to the use of orthonormal bases in curved spacetime.

□ The above argument can be made more rigorous by recognizing that the metric is a non-degenerate matrix. So, it can be diagonalized at each spacetime point through an orthogonal matrix: $E^T g E = \mathrm{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$, $E^T E = \mathrm{I}$. Re-scaling

2.9

$${e^\mu}_a := \frac{1}{\sqrt{|\lambda_a|}} {E^\mu}_a$$

gives then $e^T g e = \mathrm{diag}(1, -1, -1, -1)$. Since the metric signature $\eta = (+, -, -, -)$ is an *intrinsic property* of a manifold (Sylvester's law), this result establishes that a Lorentzian metric admits a Minkowskian base at each point of the manifold.

2.4 Orthonormality Relations

□ The *Greek* vierbein (or 'curved') indices can be raised or lowered by the metric, i.e. $g^{\mu\nu}$ or $g_{\mu\nu}$, respectively. In contrast, the *Latin* (or 'Lorentzian' or 'flat') vierbein indices refer to the tangent space; they can be manipulated using η^{ab} or η_{ab}. For example:

2.10

$$e^{\mu a} := g^{\mu\nu} e_{\nu}{}^{a} \qquad e_{\nu a} := \eta_{ab} e_{\nu}{}^{b} \qquad e^{\mu}{}_{a} := \eta_{ab} g^{\mu\nu} e_{\nu}{}^{b}$$

The vierbein field itself can be similarly manipulated: $e^{\nu}{}_{a} = e^{\mu}{}_{a} e^{\nu}{}_{\mu}$, since $e^{\nu}{}_{\mu} = \delta^{\nu}_{\mu}$. [Wikipedia: Spin Connection]

☐ By appropriately raising and lowering indices and using (1.10), one can demonstrate that the vierbein fields defined by (2.6,7) satisfy the *orthonormality relations*

2.11

$$e^{\mu}{}_{a}(x) e_{\nu}{}^{a}(x) = \delta^{\mu}_{\nu} \quad e_{\mu}{}^{a}(x) e^{\mu}{}_{b}(x) = \delta^{a}_{b}$$

confirming that $e^{\mu}{}_{a}$ is indeed the inverse of the vierbein $e_{\mu}{}^{a}$, and visa versa.

☐ It is noteworthy that the vierbeins serve double purposes:

1. the vierbein serves as *components* of the coordinate basis vectors in terms of the orthonormal tetrad (2.6), and as *components* of the reciprocal tetrad in relation to the coordinate reciprocal basis.
2. the inverse vierbein acts as the *components* of the orthonormal tetrad basis vectors in terms of the coordinate basis (2.7), and as *components* of the reciprocal coordinate basis in relation to the reciprocal tetrad basis;

☐ Formally, a *linear operator* can be assigned to map the coordinate frame into the orthonormal frame, expressed as:

2.12

$$\mathbf{e}_{a} = e^{\mu}{}_{a} \mathbf{g}_{\mu} := h(\mathbf{g}_{\mu}) \quad \mathbf{g}_{\mu} = e_{\mu}{}^{a} \mathbf{e}_{a} := h^{-1}(\mathbf{e}_{a})$$

This operator represents the vierbein field. The assumption of invertibility of the map ensures that the adjoint map $\mathbf{g}^{\mu} = \bar{h}(\mathbf{e}^{a})$ is also invertible.

Chapter 3

Geometric Algebra

3.1 Geometric Tangent Space [DH,DL,JCS]

☐ The tangent vector spaces $T_x\mathcal{M}_4 = V_4(\mathbf{x})$ of the manifold $\mathcal{M}_4 = \{\mathbf{x}\}$ can be extended to *geometric tangent spaces* $GT_x\mathcal{M}_4$ at each point. The generalization involves transforming the tangent space $T_x\mathcal{M}_4$ from a vector space into a linear space structured by the Clifford algebra $\mathrm{Cl}(1,3)$. This *geometric algebra* (GA) encompasses the algebra of differential forms (exterior algebra) as a sub-algebra under the wedge product.

☐ Specifically, geometric algebra constitutes a linear space built on an inner product vector space, augmented by the operation of wedge multiplication. The construction endows the tetrad base vectors $\{\mathbf{e}_a(\mathbf{x}); a = 0,1,2,3\}$ with the *inner product rule*

3.1

$$\mathbf{e}_a \cdot \mathbf{e}_b := \frac{1}{2}(\mathbf{e}_a\mathbf{e}_b + \mathbf{e}_b\mathbf{e}_a) = \frac{1}{2}\{\mathbf{e}_a,\mathbf{e}_b\} = \eta_{ab}$$

This means that the Clifford algebra $\mathrm{Cl}(1,3)$ of the base vectors is *isomorphic* to the algebra of Dirac matrices. However, the triad base vectors in (3.1) are not matrices, but are to be regarded as four separate vectors with a clear geometric meaning in the tangent space $GT_x\mathcal{M}_4 := \mathcal{G}_4$.

☐ In addition to (3.1), the *wedge product* is defined by:

3.2

$$\mathbf{e}_a \wedge \mathbf{e}_b := \frac{1}{2}(\mathbf{e}_a\mathbf{e}_b - \mathbf{e}_b\mathbf{e}_a)$$

The dot (inner) product of vectors is a *scalar*, whereas the wedge product of two vectors is a new type of object called a *2-vector* or *bi-vector*.

Axioms Geometric Algebra

a. The geometric product of vectors is *associative*:

$$\mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c} = \mathbf{abc}$$

b. The geometric product of vectors is *distributive* over addition:

$$\mathbf{a}(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$$

c. The square of any vector is a *real scala*r: $\mathbf{a}^2 \in \Re$

3.2 Geometric Product Rule [DL,DR,PDK]

□ With the definitions of the inner product (3.1) and outer product (3.2), the multiplicative properties of the tetrad base $\{\mathbf{e}_a\}$ can then be summarized in the *geometric product rule*

3.3

$$\mathbf{e}_a\mathbf{e}_b = \mathbf{e}_a \cdot \mathbf{e}_b + \mathbf{e}_a \wedge \mathbf{e}_b$$

An essential feature of the geometric product is that it is *invertible*, that is, the reciprocal basis $\{\mathbf{e}^a(x)\}$ may be obtained algebraically. Since a basis vector $\mathbf{e}^a$ must be orthogonal to all basis vectors $\{\mathbf{e}_b, b \neq a\}$, one derives:

3.4

$$\mathbf{e}^a := (\mathbf{e}_a)^{-1}, \quad \mathbf{e}^a\mathbf{e}_b = \mathbf{e}^a \cdot \mathbf{e}_b = \delta^a_b$$

This is an equally valid basis for vectors and tensors in the geometric tangent space $\mathcal{G}_4$.

□ Invertibility (3.4) relies on the fact that the newly defined geometric product (3.3) has the property of being *associative*, i.e. $\mathbf{a}(\mathbf{bc}) = (\mathbf{ab})\mathbf{c}$, in contrast to the inner product (3.1) and the outer product (3.2), separately. This also is key to defining more complex products such as $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$. Useful identities valid in the GA formalism are

3.5

$$\mathbf{a}(\mathbf{b} \wedge \mathbf{c}) = 2(\mathbf{a} \cdot \mathbf{b})\mathbf{c} - 2(\mathbf{a} \cdot \mathbf{c})\mathbf{b}$$

3.6

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b}$$

3.7

$$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$$

By successively multiplying together vectors the complete algebra may be generated. Elements of this algebra are called multivectors. Proofs may be found in the references indicated above.

3.3 Volume Element [DL,PDK]

☐ In geometric algebra, a *p-vector* (or *p-blade*) is the wedge product of p distinct basis vectors $\mathbf{e}_1 \wedge \mathbf{e}_2 .... \wedge \mathbf{e}_p$, or more generally, the wedge product of p linearly independent vectors. For $p = 0, 1, 2, 3$, p-vectors are called, respectively, *scalars*, *vectors*, *bivectors* and *tri-vectors*; they are dual to 0-forms, 1-forms, 2-forms and 3-forms; see appendix Duality Operation. Geometrically, a p-vector represents a p-dimensional *oriented volume* of space specified by the vectors that define its edges. Whenever two of the constituent vectors are exchanged, the volume changes sign, corresponding to a volume of *opposite orientation*.

☐ The wedge product of the *maximum number* of independent vectors of a n-dimensional space $\mathcal{G}_n$ forms a one-dimensional subspace. Its members are called *n-volumes* or *pseudoscalars*; *top forms* or *volume forms* in the language of differential forms. Similar to scalars, pseudoscalars do not have any direction aside from their *sign*.

Unit Volume

a. As the *highest-grade* element in a vector space, a pseudoscalar volume is unique up to a sign and scaling. In orientable spaces, it is possible to define a *unit volume element*.

b. In particular, the unit volume element of the tangent space $\mathcal{G}_4$ spanned by a basis $\{\mathbf{e}_j\}$ is the grade-4 pseudoscalar:

$$\mathrm{I}_4 := \mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 / \, \|\mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\| \qquad \|\mathrm{I}_4\| = 1$$

its (positive) *orientation* being specified by the ordering of the base vectors.

c. The *modulus* in the denominator may be calculated as

$$\|\mathbf{e}_0 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3\| = \ |\det[\mathbf{e}_i \cdot \mathbf{e}_j]|^{1/2}$$

d. The unit volume element has an *inverse* I^4. If the basis is orthonormal:

$$\mathrm{I}^4 = \ \mathbf{e}^0 \wedge \mathbf{e}^1 \wedge \mathbf{e}^2 \wedge \mathbf{e}^3 \qquad \mathrm{I}^4 = -\mathrm{I}_4 \qquad (\mathrm{I}_4)^2 = -1$$

☐ One usually assumes that the unit volume element is *continuous* and *differentiable* over the entire manifold, that it has the *same grade* everywhere, and that it is *single-valued*. The last assumption implies that the manifold is orientable, and so rules out objects such as the Mobius strip, where the unit volume element is double-valued.

3.4 Vector Derivative [DL,PDK]

☐ The main derivative operator in GA is the *vector derivative* $\nabla = \mathbf{e}^j \partial_j$, with $\{\mathbf{e}^j\}$ the reciprocal base with respect to an arbitrary frame field $\{\mathbf{e}_j\}$.

Vector Derivative ∇

a. yields the *gradient* when operating on a scalar field $\phi = \phi(x)$

$$\nabla\phi(x) = \mathbf{e}^j\,\partial_j\,\phi(x)$$

b. yields the *geometric product* when operating on a vector field

$$\nabla\mathbf{a} = \nabla\cdot\mathbf{a} + \nabla\wedge\mathbf{a}$$

c. yields the *divergence* through the inner product

$$\nabla\cdot\mathbf{a} = \mathbf{e}^j\cdot\partial_j\,\mathbf{a} = \partial_j\,a^j \qquad a^j := \mathbf{e}^j\cdot\mathbf{a}$$

d. yields the *curl* through the wedge product

$$\nabla\wedge\mathbf{a} = \mathbf{e}^i\wedge\,\partial_i\,\mathbf{a} = \partial_i\,a^j\,\mathbf{e}^i\wedge\mathbf{e}_j$$

□ The operator ∇ has all the *algebraic properties* of a vector and can act over any multivector with the inner, outer or geometric product. For example, if ϕ is a scalar field and $\mathbf{a}$ and $\mathbf{b}$ two vector fields, then

3.8

$$\nabla(\phi\mathbf{a}) = (\nabla\phi)\mathbf{a} + \phi(\nabla\mathbf{a})$$

3.9

$$\nabla(\mathbf{a}\wedge\mathbf{b}) = (\nabla\mathbf{a})\wedge\mathbf{b} - \mathbf{a}\wedge(\nabla\mathbf{b})$$

The commutativity or anticommutativity $(-1)^p$ depends on the degree/rank p of the object that ∇ is acting upon. For a summary of properties see references above.

3.5 Vector Coderivative [DH,JCS]

□ In geometric algebra, the notion of a *coderivative* (covariant derivative operator) can be extended to a *multivector directional derivative* (MDD), i.e. an operator which takes the derivative of a multivector field in the direction of a given vector. It can be shown that such an operator exists for every *metric-compatible* connection.

□ The basic properties of such a *directional coderivative* D_i with respect to an arbitrary frame field $\{\mathbf{e}_j\}$ may be summarized as follows:

Directional Coderivative D_i

a. maps *scalars* into scalars: $D_i\phi = \partial_i\phi$; partial derivatives operate only on *scalar components* relative to the basis $\{\mathbf{e}_j\}$;
b. maps *vectors* into vectors. In particular, D_i maps $\mathbf{e}_j$ into a vector, which can be expressed as a linear combination:

$$D_i\mathbf{e}_j = \Gamma^k_{ij}\mathbf{e}_k \quad \Gamma^k_{ij} = (D_i\mathbf{e}_j)\cdot\mathbf{e}^k$$

This defines the *connection coefficients* Γ^k_{ij} for the frame $\{\mathbf{e}_j\}$, which can be arbitrary, save for the '*metric compatibility*' condition (1.28);

c. obeys the *Leibniz rule*. Thus, for any two multivector fields A, B,

$$D_i(AB) = (D_iA)B + A(D_iB)$$

d. is *distributive* with respect to addition,

$$D_i(A + B) = D_iA + D_iB$$

e. is related to operators $D_m = h^i_m D_i$ corresponding to different frames $\{\mathbf{e}^m\}$, if the frame $\{\mathbf{e}^i\}$ is related to the different frames by $\mathbf{e}^i = h^i_m \mathbf{e}^m$.

☐ The *vector coderivative* is defined as $D := \mathbf{e}^i D_i$, with properties directly derived from the multivector directional coderivative defined above. This includes the important property that the vector coderivative may be decomposed into the sum

3.10

$$DA = D \cdot A + D \wedge A$$

of the *covariant divergence* $D\cdot A := \mathbf{e}^i \cdot D_iA$ and the *covariant curl* $\mathrm{D}A := D\wedge A := \mathbf{e}^i \wedge D_iA$. The vector coderivative is a *coordinate-free* object that can be decomposed in any system of coordinates, e.g. $D = \mathbf{g}^\mu D_\mu$. Algebraically it may be treated as a *vector*, to be used as such in geometric, inner and wedge products.

3.6 Integration Measure [DL,DH]

☐ The *line integral* of a multivector field on a manifold $\mathcal{M}_n$ along a one-parameter curve $\mathbf{x}(\lambda)$ is defined by

3.11

$$\int d\lambda \frac{\partial \mathbf{x}}{\partial \lambda} \mathbf{F}(x) := \int d\mathbf{x}_\lambda \, \mathbf{F}(x)$$

The key concept here is the *vector-valued measure*

3.12

$$d\mathbf{x}_\lambda := d\lambda \frac{\partial \mathbf{x}}{\partial \lambda} = d\lambda \left\| \frac{\partial \mathbf{x}}{\partial \lambda} \right\| \mathbf{I} \qquad \mathbf{I} := \frac{\partial \mathbf{x}}{\partial \lambda} \left\| \frac{\partial \mathbf{x}}{\partial \lambda} \right\|^{-1}$$

with I the unit vector giving an *orientation* to the domain of integration.

☐ Definition (3.12) extends naturally to higher dimensional surfaces and volumes:

Directed Measure

a. Let $\{\mathbf{g}_1, \dots, \mathbf{g}_p\}$, $1 \leqslant p \leqslant n$, be a subset of coordinate base vectors that span a p-dimensional subspace in an overall n-dimensional vector space $\mathcal{G}_n$.

b. In GA the p-blade $\mathbf{g}_1 \wedge \mathbf{g}_2 \wedge \cdots \wedge \mathbf{g}_p$ is interpreted to be the oriented volume of the p-parallelotope subtended by these basis vectors. The *oriented measure* is then constructed by defining

$$d\mathbf{x}_p := dx^1 ... dx^p \mathbf{g}_1 \wedge .... \wedge \mathbf{g}_p$$

where the $\{dx^i\}$ are ordinary scalar coordinate integration elements.

c. The normalized wedge product of the tangent vectors defines the *unit volume element* for the subspace:

$$\mathbf{I}_p(x) := \mathbf{g}_1 \wedge ... \wedge \mathbf{g}_p / \left\| \mathbf{g}_1 \wedge ... \wedge \mathbf{g}_p \right\|$$

The *orientation* is specified by the ordering of tangent vectors.

d. The case $p = n = 4$ is especially important; see Volume Element. For a metric space the denominator is the square root of the determinant $g := \mathrm{Det}(\mathbf{g}_i \cdot \mathbf{g}_j)$ of the metric tensor, leading to the *signed* scalar measure

$$dx_4 := d^4 x \mathbf{g}_0 \wedge ... \wedge \mathbf{g}_3 = d^4 x \sqrt{|g|} \mathrm{I}_4$$

This is the *invariant volume element* as defined in general relativity, multiplied by the unit volume element I_4 giving the measure a *sign*.

Chapter 4

Spin Connection

4.1 Connection Coefficients [MTW,SC]

□ The coderivative of a tensor in the coordinate basis is given by its partial derivative plus correction terms. The same procedure applies to the tetrad basis $\{\mathbf{e}_a\} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, but the Γ-connection coefficients (1.15) are replaced by the *spin connection coefficients* denoted ${\Sigma_\mu}^b{}_a$:

4.1

$$D_\mu \mathbf{e}_a := {\Sigma_\mu}^b{}_a \mathbf{e}_b \quad D_\mu \mathbf{e}^a := -{\Sigma_\mu}^a{}_b \mathbf{e}^b$$

4.2

$${\Sigma_\mu}^b{}_a = \mathbf{e}^b \cdot (D_\mu \mathbf{e}_a) := \mathbf{e}^b \cdot \Sigma_\mu(\mathbf{e}_a)$$

The name 'spin connection' comes from the fact that this formalism can be used to take coderivatives of spinors. The generic name is '*Lorentz connection*' in recognition of the underlying Lorentz symmetry of the tetrad basis. The index convention is taken from [Wikipedia: Spin Connection].

□ By definition the spin connection coefficients are *anti-symmetric*, ${\Sigma_\mu}^{ab} = -{\Sigma_\mu}^{ba}$, in their internal indices. This can be understood as following from the requirement that the internal Minkowski metric be compatible with the coderivative:

4.3

$$0 = D_\mu \eta^{ab} := D_\mu(\mathbf{e}^a \cdot \mathbf{e}^b) = -{\Sigma_\mu}^{ab} - {\Sigma_\mu}^{ba}$$

The anti-symmetry reflects the fact that the spin connection is the generator of a Lorentz transformation for each μ.

□ Given that tetrads, at any given point, are *orthonormal* by definition, the relationship between two tetrad frames is necessarily a transformation of the group $\mathrm{SO}(1,3)$. Therefore, the basis vectors at two nearby events must be related to each other by an *infinitesimal Lorentz transformation*:

4.4

$$\mathbf{e}'_a(x+dx) = \mathbf{e}_a(x) + dx^\nu \Sigma_\nu{}^b{}_a(x)\mathbf{e}_b(x)$$

which can be thought of as a rotation in spacetime relative to the given tetrad system, keeping *lengths* and *angles* the same: $\mathbf{e}'_a(x+dx)\cdot\mathbf{e}'_b(x+dx) = \eta_{ab}$. This geometric construction is called Parallel Transport.

4.2 Connection Bivector [DH,PDK]

☐ In GA, the generators of tetrad Lorentz transformations are bivectors. This statement may be explicated by introducing the *spin connection bivector*

4.5

$$\boldsymbol{\omega}_\mu := \frac{1}{2}\Sigma_{\mu ab}\mathbf{e}^a \wedge \mathbf{e}^b$$

which has the spin (Lorentz) connection coefficients as its components. In terms of this bivector, the connection relation (4.1) may be written in the form

4.6

$$D_\mu \mathbf{e}_a = \frac{1}{2}[\boldsymbol{\omega}_\mu, \mathbf{e}_a] = \boldsymbol{\omega}_\mu \cdot \mathbf{e}_a$$

It is a general theorem in GA that the commutator of a bivector with a vector is their inner product. For the proof one may use the identities (3.6)(3.7).

☐ Equation (4.6) describes a set of orthonormal basis vectors that are rotated from point to point in spacetime. Because the connection coefficients $\Sigma_{\mu ab}$ map vectors to bivectors, there can only be $4\times 6 = 24$ degrees of freedom, corresponding to 3 rotations with generators $\mathbf{e}^i \wedge \mathbf{e}^j$, and 3 boosts with generators $\mathbf{e}^0 \wedge \mathbf{e}^j$, in each of the 4 possible directions of displacement $\mathbf{g}_\mu$. Thus, the connection bivector has a clear geometric meaning as the *generator of the Lorentz transformations* displacing a set of orthonormal basis vectors in a particular direction. Its value corresponds to the angular velocity of rotation.

☐ The replacement $\partial_\mu \to \partial_\mu + \boldsymbol{\omega}_\mu$ in curved spacetime suggests a similarity between the spin connection and a gauge field in gauge theories. The gauge transformations in the case of general relativity would then correspond to the *symmetry group* of local orthogonal rotations (Lorentz transformations) that express the equivalence of observers associated with the tetrad frames; see chapter VI, Gauge Principle.

4.3 Connection Relationships [SC,JH]

☐ The coderivative of a tensor is given by its partial derivative plus connection terms. In the case of a tetrad (non-coordinate) basis, each Latin index gets a factor of the spin connection. For example, if $\mathbf{A}$ is a vector

4.7

$$D_\mu \mathbf{A} = D_\mu (A^a \mathbf{e}_a) = (\partial_\mu A^a + \Sigma_\mu{}^a{}_b A^b)\mathbf{e}_a := (D^\Sigma_\mu A^a)\mathbf{e}_a$$

Here D^Σ_μ is a short-hand notation for the coderivative with respect to the spin connection in the tensor formalism of GRT.

☐ Since a vector is a *geometric object*, it is independent of the way it is written. So, equation (4.7) must be equivalent to the coordinate representation (1.17). The relationship between connections is given by the vierbeins. From the definitions (1.15) and (4.2) of the connections one derives the *equivalent* relationships

4.8

$$\Gamma^\lambda_{\mu\nu} := \mathbf{g}^\lambda \cdot (D_\mu \mathbf{g}_\nu) = e^\lambda{}_a(\partial_\mu e_\nu{}^a + \Sigma_\mu{}^a{}_b\, e_\nu{}^b) := e^\lambda{}_a D^\Sigma_\mu e_\nu{}^a$$

4.9

$$\Sigma_\mu{}^a{}_b := \mathbf{e}^a \cdot (D_\mu \mathbf{e}_b) = e_\nu{}^a(\partial_\mu e^\nu{}_b + \Gamma^\nu_{\mu\lambda} e^\lambda{}_b) := e_\nu{}^a D^\Gamma_\mu e^\nu{}_b$$

☐ The Levi-Civita connection coefficients Γ are *symmetric* in their lower indices. In that case, these indices on the right-hand side of (4.8) need to be symmetrized. This also implies that relation (4.9) should be taken as defining the *torsion-free spin connection*, since the the Levi-Civita connection is the unique metric compatible, torsion-free connection on a Lorentzian (pseudo-Riemannian) manifold. In general, there is no restriction: the connections in (4.8,9) may also contain torsion; see Riemann-Cartan Geometry.

☐ Equation (4.9) is covariant under *manifold diffeomorphisms* (MDs), but not *local Lorentz transformations* (LLTs) $e_\mu{}^a(x) \to e'_\mu{}^a(x) = \Lambda^a{}_b(x) e_\mu{}^b(x)$. Under the latter, the spin connection transforms in-homogeneously as

4.10

$$\Sigma'_\mu{}^a{}_b = \Lambda^a{}_c\, \Sigma_\mu{}^c{}_d\, \Lambda_b{}^d + \Lambda^a{}_c\, \partial_\mu\, \Lambda_b{}^c$$

with two distinct contributions: (first term) *non-inertial* effects induced by the new frame; (second term) *inertial* contributions due to the rotation of the new frame with respect to the previous one. This results in the proper transformation of the coderivative. On the other hand, equation (4.8) is covariant under LLTs, but not MDs.

4.4 Snygg Relation [JS,PDK]

☐ In the *torsionless* case, it is possible to derive an expression for the spin connection bivector (4.5) in terms of the metric and the vierbein. The steps involved are:

1. substitute expression (4.9) into definition (4.5) of the spin connection bivector;
2. replace $\mathbf{e}^a \wedge \mathbf{e}^b \to e_\mu{}^a e_\nu{}^b \mathbf{g}^\mu \wedge \mathbf{g}^\nu$;
3. use expression (1.29) for the Levi-Civita connection coefficients.

This then yields the useful relation

4.11

$$2\omega_\mu = \mathbf{g}^\nu \wedge \nabla g_{\mu\nu} + \mathbf{g}_\nu \wedge \partial_\mu \mathbf{g}^\nu \quad \mathbf{g}^\nu = e^\nu{}_b \mathbf{e}^b$$

first derived by John Snygg. Note that, in the calculation of connection coefficients, the rule is that spacetime derivatives do *not* act on base vectors, only on scalar components relative to base vectors; see GA Coderivative, item a.

☐ A simplification occurs in the case of a *diagonal metric*: $g_{\mu\mu} = (g_{00}, g_{11}, g_{22}, g_{33})$. If then the tetrad frame is *aligned* with the coordinate frame, the vierbein is also diagonal (no summation of μ)

4.12

$$g_{\mu\mu} = \eta_\mu (e_\mu{}^\mu)^2 \quad \eta_\mu := \eta_{\mu\mu}$$

with $\eta_\mu = (1, -1, -1, -1)$ the *signature indicator*. The second term in relation (4.11) then vanishes

4.13

$$\mathbf{g}_\nu \wedge \partial_\mu \mathbf{g}^\nu = \mathbf{g}_\nu \wedge (\partial_\mu e^\nu{}_\nu)\mathbf{e}^\nu = g_{\nu\nu} e^\nu{}_\nu (\partial_\mu e^\nu{}_\nu)\mathbf{e}^\nu \wedge \mathbf{e}^\nu = 0$$

and equation (4.11) reduces for diagonal metrics to

4.14

$$2\omega_\mu = \mathbf{g}^\mu \wedge \nabla g_{\mu\mu} = \mathbf{g}^\mu \wedge \mathbf{g}^\nu \partial_\nu g_{\mu\mu}$$

without sum over μ. This formula will be applied to the Schwarzschild metric in Section VIII, Connection Coefficients.

4.5 Tetrad Postulate [SC,JY]

☐ A bit of manipulation allows to write the relations (4.8),(4.9) as the vanishing of the coderivative of the vierbeins

4.15

$$D_\mu e_\nu{}^a := \partial_\mu e_\nu{}^a - \Gamma^\lambda_{\mu\nu} e_\lambda{}^a + \Sigma_\mu{}^a{}_b e_\nu{}^b = 0$$

4.16

$$D_\mu e^\nu{}_a := \partial_\mu e^\nu{}_a + \Gamma^\nu_{\mu\lambda} e^\lambda{}_a - \Sigma_\mu{}^b{}_a e^\nu{}_b = 0$$

Either one of these equations is known as the *tetrad postulate*. However, they are just a restatement of the relation between the Levi-Civita and the spin connections and express the fact that the total covariant derivative of the vierbein with respect to *both* spacetime and tetrad indices is vanishing. It is only a postulate in the sense that geometric objects are supposed to be independent of their coordinates and basis elements.

☐ From the tetrad postulate immediately follows the *metric compatibility* of the metric tensor expressed as the vanishing of the coderivative:

4.17

$$D_\mu g_{\nu\lambda} = D_\mu(e_\nu{}^a e_\lambda{}^b \eta_{ab}) = (D_\mu e_\nu{}^a) e_\lambda{}^b \eta_{ab} + e_\nu{}^a (D_\mu e_\lambda{}^b)\eta_{ab} = 0$$

Thus, metric compatibility is equivalent to the anti-symmetry of the spin connection in its Latin indices. Stated differently, metric compatibility holds *if and only if* the connection is *Lorentzian.*

☐ The tetrad postulate also implies that one can switch index types through the coderivative, e.g.

4.18

$$D_\mu A^a = D_\mu(e_\nu{}^a A^\nu) = e_\nu{}^a D_\mu A^\nu$$

This important property is frequently used in calculations with tensor components.

4.6 Ricci Rotation Coefficients [MTW,PDK]

☐ By defining $\partial_a := e^\mu{}_a \partial_\mu$, $D_a := e^\mu{}_a D_\mu$, $\boldsymbol{\omega}_a := e^\mu{}_a \boldsymbol{\omega}_\mu$, one may write the definition of the spin connection (4.1) in the alternative forms

4.19

$$D_a \mathbf{e}_b = e^\nu{}_a \Sigma_\nu{}^c{}_b \mathbf{e}_c \;= \omega_a{}^c{}_b \mathbf{e}_c \;= \boldsymbol{\omega}_a \cdot \mathbf{e}_a$$
$$D_a \mathbf{e}^b = -\omega_a{}^b{}_c \mathbf{e}^c \;= -\boldsymbol{\omega}_a \cdot \mathbf{e}^b$$

☐ The tetrad connection coefficients

4.20

$$\omega_{abc} := e^\nu{}_a \Sigma_{\nu bc}$$

are known as Ricci connection coefficients or *Ricci rotation coefficients.* They measure the relative rotation of frame tetrads when moved in various directions, encoding thus gravitational and non-inertial effects. Equivalently, from (4.6), (4.9), (4.19), the Ricci rotation coefficients may be defined by: [Wikipedia: Christoffel symbols]

4.21

$$\omega_{abc} := \mathbf{e}_b \cdot (D_a \mathbf{e}_c) = \mathbf{e}_b \cdot \boldsymbol{\omega}_a \cdot \mathbf{e}_c = e^\mu{}_a e_{\nu[b} D^\Gamma_\mu e^\nu{}_{c]}$$

☐ In addition to the spin connection bivector (4.5), by the Ricci coefficients one also defines a spin-connection 1-form:

4.22

$$\boldsymbol{\omega}^a{}_b := \mathbf{g}^\mu \Sigma_\mu{}^a{}_b = \mathbf{e}^c \omega_c{}^a{}_b \quad \boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}$$

in the mathematics of Lie groups known as the *Maurer-Cartan connection*. Its relation with the connection bivector (4.5) is through (4.6):

4.23

$$\mathbf{g}^\mu D_\mu \mathbf{e}_a := D\mathbf{e}_a = \mathbf{g}^\mu \boldsymbol{\omega}_\mu \cdot \mathbf{e}_a = \boldsymbol{\omega}^b{}_a \mathbf{e}_b$$

☐ In spite of their close correspondence, $\boldsymbol{\omega}_{ab}$ and $\boldsymbol{\omega}_\mu$ are different geometric objects: the first one is a *matrix valued 1-form* with values in the Lie-algebra of the Lorentz group $\mathrm{SO}(1,3)$, whereas the second is a *bivector*. The Maurer-Cartan connection plays a central role in the *Cartan formalism* of GRT defining both torsion and curvature. The connection bivector is the primary connection form in the GA tetrad-formalism.

4.7 First Cartan Equation [MTW,JCS,PDK]

☐ The tetrad-based *Cartan formalism* is largely equivalent to the conventional tensor calculus of Riemannian geometry and easily integrated in the GA formulation, sharing the advantage of a compact notation. The point of difference is that the Cartan theory is formulated within the framework of a Riemann–Cartan Geometry in which the notion of *torsion* is contained in an essential way. A Riemann-Cartan geometry with vanishing torsion is identical to the Riemannian geometry of general relativity.

☐ The close correspondence between the Cartan- and GA-formalisms may be illustrated by considering the torsion 2-form (1.24). The tetrad postulate (4.15) allows the replacement of the Levi-Civita connection by the spin (Lorentz) connection:

4.24

$$\mathbf{T}^a := e_\kappa{}^a \mathbf{T}^\kappa = \left(\partial_\mu e_\nu{}^a + \Sigma_\mu{}^a{}_b e_\nu{}^b\right) \mathbf{g}^\mu \wedge \mathbf{g}^\nu$$

The term with the spin connection at the right-hand side can be rewritten in terms of two different wedge 2-forms

4.25

$$-\omega_{[bc]}{}^a \mathbf{e}^b \wedge \mathbf{e}^c = \boldsymbol{\omega}^a{}_b \wedge \mathbf{e}^b$$

build out of anti-symmetrized Ricci coefficients (4.20), left, and the Maurer-Cartan connection (4.22), right.

☐ The latter results in the *first Cartan structure equation* for the torsion 2-form

4.26

$$\mathbf{T}^a = \mathrm{d}\mathbf{e}^a + \boldsymbol{\omega}^a{}_b \wedge \mathbf{e}^b := \mathrm{D}\mathbf{e}^a$$

The operator d is the exterior derivative as defined in (1.7). The Cartan *exterior coderivative* D: $= \mathbf{g}^\mu \wedge D_\mu$ is the covariant curl, see (3.10), associated with the Maurer-Cartan connection $\boldsymbol{\omega}^a{}_b$. This derivative raises the degree of any p-form by one and transforms covariantly under local Lorentz transformations on account of (4.10).

□ In Einstein's GRT torsion vanishes. In that case the first Cartan equation reduces to the constraining equation

4.27

$$\mathrm{D}\mathbf{e}^a = \mathrm{d}\mathbf{e}^a + \boldsymbol{\omega}^a{}_b \wedge \mathbf{e}^b = 0$$

which is the Cartan equivalent of the Levi-Civita torsion constraint (1.26). This constraint is lifted in the Einstein-Cartan Theory (ECT).

4.8 Structure Coefficients [MTW,PDK]

□ Setting the torsion to zero in equation (4.24), one derives the constraint equation (4.27) in component form

4.28

$$0 = 2\left(\partial_{[\mu} e_{\nu]}{}^a + \Sigma_{[\mu|}{}^a{}_b e_{|\nu]}{}^b\right) = f_{\mu\nu}{}^a - 2e_{[\mu}{}^b e_{\nu]}{}^c \omega_{bc}{}^a$$

as an relation between the Ricci connection (4.20) and the so-called *structure coefficients* defined as the curl of the vierbeins. When expressed in the tetrad basis

4.29

$$f_{ab}{}^c := e^\mu{}_a e^\nu{}_b f_{\mu\nu}{}^c = 2e^\mu{}_a e^\nu{}_b \partial_{[\mu} e_{\nu]}{}^c$$

these are also known as *coefficients (or objects) of anholonomy* because they specify how much the tetrad frame $\{\mathbf{e}_a\}$ departs from being holonomic, i.e., $\partial_{[\mu} e_{\nu]}{}^c = 0$, as is the case for the coordinate basis $\{\mathbf{g}_\mu\}$ being a gradient basis; see (1.25). In this special case, in the absence of torsion, the Ricci connection is symmetric in its two first indices.

□ With (4.29) it follows that equation (4.28) is satisfied if and only if:

4.30

$$2\omega_{[ab]c} = f_{abc}$$

By adding the same equation with a cyclic permutation of the free indices $abc \to cab$ and then subtracting a further cyclic permutation, the Ricci connection emerges as a pure combination of the anholonomy coefficients.:

4.31

$$\omega_{abc} = \frac{1}{2}\left(f_{abc} + f_{cab} - f_{bca}\right)$$

The anti-symmetry of the Ricci connection in its *last two indices* is ensured by the anti-symmetry of the structure coefficients in their *first two indices*.

□ Thus, the (torsionless) spin connection is *completely determined* by the vierbeins. One may note the similarity to the Christoffel formula (1.29) for Levi-Civita connection coefficients. This is not by coincidence because it is a theorem that there exists a unique metric torsion-free connection which is the one given by (1.29) or, given a triad form, (4.31).

Chapter 5

Parallel Transport

5.1 Tangent Vectors [MTW,DL]

□ Consider a smooth *spacetime curve* $\mathbf{x}(\lambda)$ parameterized by a parameter λ. The *tangent vectors* to this curve are given by

5.1

$$\mathbf{u}(\lambda) := \frac{d\mathbf{x}}{d\lambda} = \frac{\partial \mathbf{x}}{\partial x^\mu}\frac{dx^\mu}{d\lambda} = u^\mu \partial_\mu \mathbf{x} = u^\mu(\lambda)\mathbf{g}_\mu(\mathbf{x}(\lambda))$$

with respect to the coordinate basis along the curve.

□ For a vector $\mathbf{Y}(\lambda) := \mathbf{Y}(\mathbf{x}(\lambda))$, the *directional coderivative* along the curve is defined as

5.2

$$\frac{D\mathbf{Y}}{d\lambda} := u^\mu D_\mu \mathbf{Y} = \mathbf{u} \cdot D\mathbf{Y}$$

This is a natural generalization of $D_\mu := \mathbf{g}_\mu \cdot D$, the coderivative along the basis vector $\mathbf{g}_\mu$. In the literature the directional derivative $\mathbf{u} \cdot D$ is often denoted as (bold) $\nabla_{\mathbf{u}}$.

□ Any vector $\mathbf{Y}$ can be expanded in terms of the basis vectors of the *coordinate frame* as $\mathbf{Y} = Y^\nu \mathbf{g}_\nu$. In view of (1.17) the coordinate form of (5.2) is then

5.3

$$u^\mu D_\mu \mathbf{Y} = u^\mu(\partial_\mu Y^\nu + \Gamma^\nu_{\mu\kappa} Y^\kappa)\mathbf{g}_\nu$$

If the same vector is expanded with respect to the *local tetrad frame* $\{\mathbf{e}_a\}$, the connection is determined by the bivector (4.5):

5.4

$$u^\mu D_\mu \mathbf{Y} = (u^\mu \partial_\mu Y^a)\mathbf{e}_a + u^\mu \boldsymbol{\omega}_\mu \cdot \mathbf{Y}$$

5.2 PT Equation [MTW,DL]

□ A vector $\mathbf{Y}(\lambda)$ is said to be *parallel transported* along $\mathbf{x}(\lambda)$, if the coderivative along the curve vanishes

5.5

$$\mathbf{u} \cdot D\mathbf{Y} = 0$$

In view of (5.3) the coordinate form of the *PT equation* (5.5) is

5.6

$$\left(\frac{dY^\nu}{d\lambda} + \frac{dx^\mu}{d\lambda} \Gamma^\nu_{\mu\kappa} Y^\kappa \right) \mathbf{g}_\nu = 0$$

If vectors are decomposed with respect to a tetrad frame, the PT equation takes the form:

5.7

$$\mathbf{u} \cdot \nabla\mathbf{Y} + \boldsymbol{\omega}(u) \cdot \mathbf{Y} = 0$$

with $\boldsymbol{\omega}(u) := u^\mu \boldsymbol{\omega}_\mu$ and $\mathbf{u} \cdot \nabla = u^\mu \partial_\mu$ the directional derivative which acts on the components Y^a relative to the tetrad frame.

□ Thus, a vector $\mathbf{Y}(\lambda)$ is parallel-transported in the $\mathbf{u}$-direction, if the variation in the vector components is *precisely* compensated by the transformation of the transported frame along the given curve. The importance of this procedure is that in this way vectors and tensors that belong to *different* tangent spaces can be compared with each other. It is worth emphasizing that the affine connection structure of the spacetime manifold is crucial to this purpose.

□ The equation of parallel transport is a first-order differential system in the unknowns $Y^\mu(\lambda)$ or $Y^a(\lambda)$. When the initial vector $\mathbf{Y}(\lambda_0)$ has been specified by the initial data, there exists a unique solution of the equations (5.6), or (5.7), along the curve of transport. The solution depends on the curve $\mathbf{x}(\lambda)$. Parallel transport is also dependent on the connection, and different connections lead to different answers. If the connection is metric-compatible, see (1.28), the metric is always parallel transported: $\mathbf{u} \cdot Dg_{\mu\nu} = 0$. It follows that the inner product of two parallel-transported vectors is preserved as is their norm, i.e. parallel transport is an *isometry*.

5.3 Geodesics [MTW]

□ A curve $\mathbf{x}(\lambda)$ is said to be *auto-parallel* or a *geodesic curve* if its tangent vector satisfies the condition

5.8

$$u^\mu D_\mu \mathbf{u} = \mathbf{u} \cdot D\mathbf{u} = 0$$

That is, if it remains parallel to itself along $\mathbf{x}(\lambda)$. In GRT, a geodesic curve generalizes the notion of a 'straight line' to curved spacetime, namely, by being as straight as possible.

☐ If the parameter λ is changed $\lambda \to \tau$, the tangent vector changes:

5.9

$$\mathbf{u}(\tau) = \frac{d\mathbf{x}}{d\lambda}\frac{d\lambda}{d\tau} = \mathbf{u}(\lambda)\frac{d\lambda}{d\tau}$$

The auto-parallel character of the curve is *not modified* by this parameter change if and only if the parameter is *affine*, that is, $\lambda = a\tau + b$, where a, b are arbitrary constants. This linear scaling of the path amounts to the freedom to change the unit of length/time and to choose any initial point.

☐ Geodesics may be characterized as a path of *extremal length* between two given spacetime points $[\mathbf{x}_1, \mathbf{x}_2]$. From the line element $ds^2 = d\mathbf{x} \cdot d\mathbf{x}$, one defines the length of a curve $\mathbf{x}(\lambda)$ between the given points by the *proper lenght integral*

5.10

$$s[\mathbf{x}(\lambda)] := \int_1^2 ds = \int_1^2 \sqrt{|d\mathbf{x} \cdot d\mathbf{x}|} = \int_1^2 \sqrt{|\mathbf{u} \cdot \mathbf{u}|}d\lambda$$

where the integral is over the path. If the manifold $\mathcal{M}_4$ is connected, any two spacetime points can be connected by a smooth curve. The geodesic is then that curve that *extremizes* the proper length s. From definition (5.10) it is seen that this result depends neither on the choice of coordinates nor on the parametrization of the curve.

5.4 Geodesic Equation [MTW]

☐ If a geodesic curve is *timelike*, it is a possible world line for a particle. Its uniformly ticking parameter $\lambda = a\tau + b$ is a multiple of the particle's *proper time* defined by normalizing the tangent vector $\mathbf{v} := d\mathbf{x}/d\tau$, $\mathbf{v} \cdot \mathbf{v} = 1$. In a spacetime with Lorentzian metric the timelike/null/spacelike character of the geodesic (relative to a metric-compatible connection) never changes and timelike geodesics are maxima of the proper time.

☐ In the *absence* of gravity, the equation of motion of a free particle is given by:

5.11

$$\frac{d\mathbf{v}}{d\tau} = \frac{\partial \mathbf{v}}{\partial x^\mu}\frac{dx^\mu}{d\tau} = v^\mu \partial_\mu \mathbf{v} = 0$$

In the *presence* of gravity this becomes an equation that nullifies the *proper acceleration*

5.12

$$\mathbf{a} := \frac{D\mathbf{v}}{d\tau} := v^\mu D_\mu \mathbf{v} = \mathbf{v} \cdot D\mathbf{v} = 0$$

That is, in GR a ‘freely falling’ particle follows a geodesic which is a curve that *parallel-transports* its tangent vector $\mathbf{v} = d\mathbf{x}/d\tau$ along itself, preserving the norm $\mathbf{v} \cdot \mathbf{v} = 1$.

☐ To calculate the geodesic, one may use the component formalism, in particular equation (5.6). Equation (5.12) then gives the *geodesic equation* describing a (point) particle in free fall

5.13

$$\frac{dv^\mu}{d\tau} + \Gamma^\mu_{\nu\kappa} v^\nu v^\kappa = 0$$

It is a second-order differential system in the unknowns $x^\mu(\tau)$ involving coordinate rates of a change of the metric tensor describing gravity. The equation can be solved (in principle) when initial data have been specified.

5.5 Newtonian Limit

☐ In the limit of a slowly moving particle in a *weak stationary* gravitational field, the geodesic equation (5.13) reduces to the Newtonian form. This limit involves three approximations:

1. **weak field**: deviations from the Minkowskian metric $h_{\mu\nu}(x) := g_{\mu\nu}(x) - \eta_{\mu\nu}$ are small, so that only terms linear in $h_{\mu\nu}$ need be retained;
2. **stationary field**: the deviations are independent of time, $h_{\mu\nu}(t, \vec{x}) = h_{\mu\nu}(\vec{x})$;
3. **non-relativistic limit**: velocities are much smaller than the speed of light: $v \ll c$, $v^2/c^2 \leqslant h_{\mu\nu}$.

☐ The dominant term of the metric in this approximation is $g_{00} = 1 + h_{00}$, which leads to Newtons equation of motion:

5.14

$$\frac{d^2\mathbf{x}}{dt^2} = -\frac{1}{2}c^2 \nabla h_{00} \qquad h_{00} \simeq 2\frac{\Phi}{c^2} + \text{const}$$

Here Φ is the *Newtonian gravitational potential*, assumed to be small: $\Phi \ll c^2$.

☐ The additive constant to the potential is usually defined in such a way that Φ vanishes when g_{00} assumes its Minkowski value η_{00}. It is worthwhile remarking that, to the level of approximation adopted, only g_{00} enters the equation of motion, although the deviations of the other metric coefficients may be of the same order of magnitude. It is this circumstance that allows to describe non-relativistic gravity, to a first order, by means of a *single scalar* potential.

☐ This result might suggest that the connection term in the geodesic equation (5.13) is a representation of the *gravitational field* in GRT. Einstein strongly opposed this view because the whole left side of the geodesic equation (5.13) is *tensorial* (with respect to arbitrary coordinate transformations), whereas the two terms separately are not. Hence, gravity will always be intertwined with inertia, which is no surprise since locally they are indistinguishable. For this reason Einstein was of the opinion “that there is no need whatsoever to distinguish ‘inertial terms’ and ‘gravitational terms’ in the geodesic equation”. [The Princeton lectures (1921), published as the ‘The Meaning of Relativity’.]

☐ In GRT there is *no concept* of gravitational force. Curvature is used to geometrize the gravitational interaction. The gravitational interaction in this case is described by letting particles follow the curvature of spacetime. Geometry replaces the concept of force, and trajectories are determined, not by force equations, but by *geodesics*.

Chapter 6

Physical Geometry

6.1 Postulates [MTW,DLE,CR]

☐ The geometric formulation of General Relativity Theory (GRT) is underpinned by two fundamental postulates:

1. **Geodesic hypothesis**: the trajectory of a freely falling particle with *non-zero rest mass* is a coordinate-independent process, aligned with a *time-like geodesic worldline* of spacetime;
   the worldline of a free test particle with *zero rest mass* (such as a photon or neutrino) is represented by a *geodesic null-line* of spacetime.
2. **Einstein's equivalence principle**: at any point in spacetime, it is possible to establish a *locally inertial coordinate system* where matter adheres to the laws of *special relativity.*

☐ Ad 1: Einstein's geodesic law of motion may be perceived as an extension Galileo's law of inertia: *free particles invariably traverse geodesics dictated by the metric, regardless of whether spacetime is flat or curved.*
Through later work it now seems to be established that, in fact, the geodesic hypothesis is not a separate postulate, but a theorem under very general conditions concerning the equations of motion governing matter, including Einstein's field equations; see Einstein Equation.

☐ Ad 2: In a letter to Paul Painlevé dated December 7, 1921, Einstein elaborates:

> "According to the special theory of relativity the coordinates (t, x, y, z) are directly measurable via clocks at rest with respect to this coordinate system. Thus, the invariant ds, which is defined via the equation $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$, likewise corresponds to a measurement result."
> "The general theory of relativity rests entirely on the premise that each infinitesimal line element of the spacetime manifold physically behaves like the

four-dimensional manifold of the special theory of relativity. Thus, there are infinitesimal coordinate systems (inertial systems) with the help of which the *ds* are to be defined exactly like in the special theory of relativity."

☐ Through the equivalence principle which guarantees the *local validity* of special relativity, the Minkowski metric of these local inertial systems acquires its *chronometric* significance, establishing its relationship to *measurements of length and time*.

6.2 Local Flatness [MTW,CR,KL]

☐ The crux of Einstein's equivalence principle is that, *locally*, the gravitational force can be negated by a suitable acceleration of the reference frame. Or equivalently, an observer undergoing constant acceleration may, as far as all laws of nature are concerned, consider itself at rest in a homogeneous gravitational field. Over extended distances, however, objects in free fall within a realistic gravitational field converge towards one other. This *tidal effect* is accounted for in general relativity as a consequence of the curvature of spacetime; see Chapter X.

☐ Thus, Einstein's equivalence principle is a *local principle* that is made more precise in the following theorem:

Local Flatness Theorem [S. C. Fletcher, 2023]

Given a Lorentzian spacetime $(\mathcal{M}_4, g_{\mu\nu})$, any embedded curve γ therein, and any point P. Then, there exists, within some neighborhood containing the point P, a flat metric $\eta_{\mu\nu}$, such that on the curve γ:

1. $g_{\mu\nu} = \eta_{\mu\nu}$
2. $D = \nabla$, where D and ∇ are the Levi-Civita derivative operators associated with $g_{\mu\nu}$ and $\eta_{\mu\nu}$, respectively.

The theorem does not necessitate that the curve be a geodesic or even time-like.

☐ Given a flat metric with the aforementioned properties, one can invariably find an *isometry* from the region where the metric is defined to a region of Minkowski spacetime, subsequently employing that isometry to introduce standard Minkowski coordinates. Thus, a geodesic (non-spinning) frame in $\mathcal{M}_4$ is the geometric representation of a *local observer frame of reference*.

☐ The construction cannot be done in the whole of spacetime but only in a *neighborhood* of the worldline of the observer small enough to ignore tidal effects. It can be shown that this imposes the heuristic condition $a \cdot l/c^2 \ll 1$, where a is a measure for the local acceleration and l the extension of the local neighborhood. For example, if the acceleration is of the order of $10\,\mathrm{m/s^2}$ then the length is of the order of one light year, which is big enough to encompass all laboratories and observatories in the solar system.

6.3 Gauge Principle [DH,JB]

☐ A tetrad vector (covector) base $\{\mathbf{e}_a, \mathbf{e}^a\}$ of a local tangent space is not unique, since it is always possible to find an infinity of other bases by local Lorentz transformations

6.1

$$\mathbf{e}'_a(x) := \Lambda_a{}^b(x)\mathbf{e}_b(x)$$

The equivalence of these tangent bases may be seen as a *gauge principle*, where the local gauge group is the symmetry group $\mathrm{SO}(1,3)$ of position dependent Lorentz rotations (6.1). *Gauge invariance* is then the statement that physical laws cannot depend on such transformations. [Wikipedia: Ryoyu Utiyama, 1956]

☐ From a given local frame, different classes of non-inertial frames may be obtained by performing local (point-dependent) Lorentz transformations. This includes transformations that boost away local gravity. The gauge principle of local Lorentz symmetry thus encompasses the essence of Einstein's *equivalence principle* that gravitation may be annulled by acceleration.

☐ To derive the gauge invariant equations implied by the gauge principle, one needs a *gauge covariant derivative*. To ascertain that the coderivative D_μ fulfills the necessary and sufficient conditions, one may differentiate (6.1) to get

6.2

$$D_\mu \mathbf{e}'_a(x) = \Sigma'_\mu{}^b{}_a(x)\mathbf{e}'_b(x)$$

Here Σ' is the locally transformed spin connection (4.10). This proves that the coderivative D_μ is invariant under a change of gauge, the spin (Lorentz) connection Σ_μ^{ab}, with one spacetime index and two tetrad indices, being the relevant *gauge field*.

☐ The *gauge field strength*, in gauge theory defined by the coderivative of the gauge field

6.3

$$R_{\mu\nu}^{ab} := 2D^\Sigma_{[\mu}\Sigma^{ab}_{\nu]} = \partial_\mu \Sigma_\nu^{ab} - \partial_\nu \Sigma_\mu^{ab} + \Sigma^a_{\mu c}\Sigma^{cb}_\nu - \Sigma^a_{\nu c}\Sigma^{cb}_\mu$$

turns out to be the *curvature tensor* associated with the spin connection; see equation (10.2). This field is a tensor under MDs owing to the antisymmetric derivatives, and transforms homogeneously under LLTs.

6.4 Gauge Theory [JY,JB]

☐ GRT may be viewed as a *gauge theory* of the Lorentz algebra based on the *restricted Lorentz group* $\mathrm{SO}^+(1,3)$. This is the component of the Lorentz group connected to the identity. Alternatively, it can be defined as the subgroup of transformations which are orthochronous (preserve the direction of time) and proper (preserve the orientation).

☐ For this restricted group, the representation $\Lambda^\mu{}_\nu$ of 4×4 real matrices acting on $\mathbb{R}^{1,3}$ may be put in the exponentiated form

6.4

$$\Lambda = \exp\left(\frac{1}{2}\varepsilon_{ab}M^{ab}\right)$$

The anti-symmetric group generators are 4x4 matrices that satisfy the *Lorentz algebra* commutation relations with the components

6.5

$$(M^{ab})^c{}_d := \eta^{ac}\delta^b_d - \eta^{bc}\delta^a_d$$

The anti-symmetric parameters $\varepsilon_{ab} = -\varepsilon_{ba}$ may be identified with an infinitesimal Lorentz transformation: $\Lambda_{ab} = \eta_{ab} + \varepsilon_{ab}$. Being anti-symmetric, the ε_{ab} have $4\text{x}3/2 = 6$ independent components, which agrees with the 6 transformations of the Lorentz group: 3 rotations and 3 boosts. [Wikipedia: Dirac Algebra]

☐ More generally, one may associate six anti-symmetric 4x4 matrices J^{ab} with the Lorentz algebra, to generate any 4x4 matrix representation $S(\Lambda)$ of $\mathrm{SO}^+(1,3)$ by the exponential form:

6.6

$$S(\Lambda) = \exp\left(\frac{1}{2}\varepsilon_{ab}J^{ab}\right)$$

Under local LTs, the parameters ε_{ab} become a function of spacetime: $\varepsilon_{ab} \to \varepsilon_{ab}(x)$. This step of making a symmetry local is called *gauging* the global symmetry. As a consequence, derivatives of fields like $\partial_\mu\psi(x)$, are no longer homogeneous due to the occurrence of terms like $[\partial_\mu\varepsilon_{ab}(x)]\psi(x)$, which are non-vanishing unless ε_{ab} is constant. The procedure to go from a locally flat to a curved spacetime involves the generalization of ∂_μ to a covariant derivative $\mathcal{D}_\mu$ to compensate for these extra terms.

6.5 Fock-Ivanenko Coderivatives [JY,JB]

Geometric objects may lose their covariant character under point-dependent Lorentz transformations, in particular, spacetime derivatives of covariant objects. The general procedure to restore proper tensor behavior, is to introduce linear connections. These entities belong to the subgroup $\mathrm{SO}^+(1,3)$ and are called *Lorentz connections* or spin connections.

☐ Lorentz connections may be characterized as 1-forms acting in the Lorentz algebra:

6.7

$$\boldsymbol{\omega}_\mu(x) := \frac{1}{2}\Sigma_{\mu ab}(x)J^{ab}$$

The anti-symmetric quantities $J^{ab} = -J^{ba}$ are the generators of the appropriate representation (6.6) of the Lorentz subgroup $\mathrm{SO}^+(1,3)$. The coefficients $\Sigma_{\mu ab}$, anti-symmetric in the Latin indices, are the *spin connection coefficients* defined in (4.2). These fields transform inhomogenously according to (4.9). In the present context, they may be looked upon as *gauge fields* with one spacetime index and two group indices.

□ The Lorentz connections (6.7) permit the introduction of the *Fock–Ivanenko* (FI) coderivative operator, first proposed by Vladimir Fock and Dmitri Ivanenko in 1929:

6.8

$$\mathcal{D}_\mu := \partial_\mu + \boldsymbol{\omega}_\mu = \partial_\mu + \frac{1}{2}\Sigma_{\mu ab}J^{ab}$$

The importance of the FI coderivative is that it can be defined for *all tensorial* and *spinorial* fields. This may be illustrated by the following three examples:

6.9

a. $J^{ab} = M^{ab}$: $\qquad \mathcal{D}_\mu = D_\mu = \partial_\mu + \Sigma_\mu{}^b{}_a$

With the generator (6.5), the Lorentz connection (6.7) reduces to the spin connection (4.2) as defined in the tensorial formalism of GRT.

b. $J^{ab} = \mathbf{e}^a \wedge \mathbf{e}^b$: $\qquad \mathcal{D}_\mu = D_\mu = \partial_\mu + \boldsymbol{\omega}_\mu$

The Lorentz connection (6.7) is then identical to the GA bivector (4.5).

c. $J^{ab} = \frac{1}{2}\gamma^a \wedge \gamma^b$: $\qquad \mathcal{D}_\mu = \partial_\mu + \frac{1}{4}\Sigma_{\mu ab}\gamma^a \wedge \gamma^b$

The γ's are 4x4 Dirac matrices. This form of the Lorentz connection is relevant to the Dirac equation and -theory in curved spacetime.

6.6 Dirac Equation [SW,JY]

□ In a tetrad Minkowski space $\{x^a\}$, the *Dirac equation* for a free spin-1/2 particle having mass m may be written in the covariant form

6.10

$$(i\hbar\gamma^a\partial_a - m)\psi(x) = 0 \qquad a = 0, 1, 2, 3$$

Plank's constant $\hbar$ is henceforth set equal to one. The *Dirac spinor* $\psi(x)$ is a 1x4 complex matrix field and the four *gamma matrices* $\{\gamma^a\}$ are a set of complex 4x4 matrices which satisfy the defining anti-commutation relations:

6.11

$$\gamma^a \cdot \gamma^b := \frac{1}{2}(\gamma^a\gamma^b + \gamma^b\gamma^a) = \eta^{ab}I$$

The Dirac Algebra is a Clifford algebra $\mathrm{Cl}(1,3)(\mathbb{C})$ with the gamma matrices as generators. It is conventional to suppress component indices of the gamma matrices and the spinor. Usually, the identity matrix at the right-hand side is also omitted.

☐ Under Lorentz transformations $x'^a = \Lambda^a{}_b x^b$, the Dirac spinor transforms as

6.12

$$\psi(x) \to \psi'(x) := S(\Lambda)\psi(\Lambda^{-1}x)$$

The 4x4 matrices $S(\Lambda)$ form the *spin representation* of the restricted Lorentz group $\mathrm{SO}^+(1,3)$. The transformed Dirac equation then becomes

6.13

$$S(\Lambda)\left[iS^{-1}(\Lambda)\gamma^a S(\Lambda)(\Lambda^{-1})^b{}_a\partial_b - m\right]\psi(\Lambda^{-1}x) = 0$$

The Lorentz transformation acting on the Lorentz index, commutes with internal Lorentz transformation $S(\Lambda)$ that acts on spinor objects: $[\Lambda^a{}_b, S(\Lambda)] = 0$.

☐ Between the brackets, the original Dirac operator is regained if

6.14

$$S^{-1}(\Lambda)\gamma^a S(\Lambda) = \Lambda^a{}_b\gamma^b$$

Hence, in flat space, the Dirac equation is *Lorentz invariant*, provided equality (6.14) is true. This may be shown to be the case by considering Lorentz transformations near the identity. [Wikipedia: Dirac Equation]

6.7 Spinor Connection [JY,CK]

☐ A formally covariant Dirac equation in GRT is most easily constructed by introducing spacetime-dependent gamma matrices by way of *Einstein's vierbein fields*:

6.15

$$\gamma^\mu(x) := \mathrm{e}^\mu{}_a(x)\gamma^a, \qquad \gamma_\mu(x) := \mathrm{e}_\mu{}^a(x)\gamma_a$$

On account of the basic vierbein equality (2.4), their *anti-commutator* is seen to be equal to the metric tensor:

6.16

$$\gamma^\mu \cdot \gamma^\nu := \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g^{\mu\nu}I$$

The derivative term of the Dirac equation (6.10) now can be put in the familiar form $i\gamma^\mu\partial_\mu\psi(x)$, but with spacetime-dependent gamma matrices.

☐ In curved spacetime, also the spinor transformation matrix depends on spacetime location: $S(\Lambda) \to S[\Lambda] := S[\Lambda(x)]$. As a consequence, the spacetime derivative of a spinor does not transform as a spinor. The remedy is to replace the derivative by the appropriate FI-coderivative (6.8):

6.17

$$\partial_\mu \psi(x) \to \mathcal{D}_\mu \psi(x) = (\partial_\mu + \Gamma_\mu)\psi(x)$$

The *spinor connection* Γ_μ only acts on spinor indices and has the explicit form, see (6.9c):

6.18

$$\Gamma_\mu := \frac{1}{4}\Sigma_{\mu ab}\gamma^a \wedge \gamma^b$$

☐ The corresponding Dirac equation in *curved spac*e then simply reads

6.19

$$(i\gamma^\mu \mathcal{D}_\mu - m)\psi(x) = 0$$

The coefficient matrices γ^μ are not unique. It rests to be shown that all representations are equivalent and that the Dirac equation (6.19) is invariant under local Lorentz transformations.

6.8 Lorentz Invariance [JY,CK]

☐ Since local *Lorentz invariance* is a fundamental symmetry in GRT, the Dirac equation (6.19) should be invariant under the combined local transformations (6.12) and:

6.20

$$\mathcal{D}_a \psi(x) \to (\Lambda^{-1})^b{}_a \mathcal{D}'_b \psi'(x)$$

The tetrad index of the coderivative transforms according to the Lorentz group representation $\Lambda_a{}^b(x)$, whereas the spin representation generates the spinor transformation $S[\Lambda]$.

☐ The *tetrad coderivatives* in (6.20) are $e^\mu{}_a \mathcal{D}_\mu := \mathcal{D}_a = \partial_a + \Gamma_a$ and $\mathcal{D}'_a := \partial_a + \Gamma'_a$ with

6.21

$$\Gamma_a := e^\mu{}_a \Gamma_\mu = \frac{1}{4}\omega_{abc}\gamma^b \wedge \gamma^c$$

the tetrad spinor connection (6.18) expressed in terms of the *Ricci rotation coefficients* defined in (4.20). The primed spinor connection is obtained by substituting the transformed spin connection coefficients (4.10) into (6.18). With the fundamental relation (6.14) it is then found that the tetrad spinor connections are related by the (invertible) transformation

6.22

$$\Gamma'_a = S[\Lambda](\partial_a + \Gamma_a)S^{-1}[\Lambda]$$

☐ The last property ensures *Lorentz covariance* of the coderivative:

6.23

$$\mathcal{D}'_a S[\Lambda]\psi(\Lambda^{-1}x) = S[\Lambda]\mathcal{D}_a\psi(\Lambda^{-1}x)$$

because the inhomogeneous derivative terms cancel out at the right-hand side. Analogously to (6.13,14), the Lorentz invariance of the Dirac equation in curved spacetime follows:

6.24

$$(i\gamma^a\mathcal{D}_a - m)\psi(x) \to S(\Lambda)(i\gamma^a\mathcal{D}_a - m)\psi(\Lambda^{-1}x)$$

□ The covariant Dirac equation may be viewed as a result of the *gravitational minimal coupling prescription*, that is, the coupling of matter fields to gravitation may be achieved by replacing ordinary partial derivatives in the physical laws of special relativity by Fock-Ivanenko coderivatives.

Chapter 7

Fermi-Walker Transport

7.1 Observer Tetrad [MTW]

□ Let an (ideal) observer, undergoing *arbitrary*, but smooth, motion along a timelike curve $\mathbf{x}(\lambda)$, carry an *orthonormal tetrad* of basis vector fields $\{\mathrm{e}_a(\mathbf{x}(\lambda)); a = 0, 1, 2, 3\}$; the parameter λ is the *proper time* of the observer.
The time axis is chosen as the time axis of the comoving frame in which the observer is momentarily at rest. That is, the zeroth base vector is at *each instant* identified with the 4-velocity: $\mathrm{e}_0 = \mathbf{u}$, $\mathbf{u} := d\mathbf{x}/d\lambda$. The space-like elements $\{\mathrm{e}_i; i = 1, 2, 3\}$, subject to the constraint $\mathrm{e}_i \cdot \mathbf{u} = 0$, provide a Cartesian basis $\mathrm{e}_i \cdot \mathrm{e}_j = \delta_{ij}$ in the observer rest-frame.

□ Two conditions on the moving observer frame are imposed:

1. basis vectors of the tetrad must remain *orthonormal*;
2. spatial base vectors should be kept '*nonrotating*', i.e., they should remain 'as constant as possible' subject to the constraint $\mathrm{e}_i \cdot \mathbf{u} = 0$, without additional rotation.

The tetrad $\{\mathrm{e}_a(\mathbf{x})\}$ so constructed at each spacetime point may be interpreted as constituting a *local observer frame*, to be used as a tool to extract physical, measurable quantities from geometric, coordinate-free objects in general relativity.

□ In the case that the observer follows a geodesic, the criterion for a *non-spinning* inertial frame is simply $\mathbf{u} \cdot D\mathrm{e}_i = 0$, i.e. the whole triad is parallel-transported along the worldline of the observer:

7.1

$$\frac{D\mathrm{e}_a}{d\lambda} = \mathbf{u} \cdot D\mathrm{e}_a = 0$$

Non-spinning local Lorentz frames hold a special place in GRT, because they are as close as one can get in a curved Lorentzian manifold to the *inertial frames* of special relativity.

7.2 Fermi Coordinates [MTW]

☐ Given a geodesic curve and a non-spinning tetrad frame defined by (7.1), any vector $\mathbf{Y}$ can be expanded as $\mathbf{Y}(\lambda) = Y^a(\lambda)\mathrm{e}_a(\lambda)$ in terms of the basis vectors of that frame. Such local coordinates adapted to a geodesic are called *Fermi normal coordinates*. These coordinates are the natural coordinate system of an observer in general relativistic spacetimes.

☐ The *F-derivative* of any vector function $\mathbf{Y}(\lambda)$ along a curve $\mathbf{x}(\lambda)$ is defined as the vector function given by the rate of change relative to the geodesic frame:

7.2

$$\frac{d_{\mathrm{F}}\mathbf{Y}}{d\lambda} := \frac{dY^a}{d\lambda}\mathrm{e}_a$$

In other words, the F-derivative of $\mathbf{Y}(\lambda)$ is evaluated with respect to the family of Lorentz frames, whereby the basis vectors $\{\mathrm{e}_a(\mathbf{x})\}$, which are parallel transported according to (7.1), are supposed to be *constant*, as if they represented an inertial Lorentz frame.

☐ The F-derivative (7.2) is related to the coderivative (5.2) by the formula

7.3

$$\frac{d_{\mathrm{F}}\mathbf{Y}}{d\lambda} = \frac{D\mathbf{Y}}{d\lambda} - Y^a\frac{D\mathrm{e}_a}{d\lambda} = \frac{D\mathbf{Y}}{d\lambda}$$

In other words, using these coordinates, all connection coefficients Γ *vanish exactly* on the geodesic curve $\mathbf{x}(\lambda)$. If the F-derivative of some vector field vanishes, that is, its components Y^a are constant, this vector it said to experience *Fermi transport* or to be Fermi-transported along the world line of the observer. Fermi-transport is a special instance of Fermi-Walker transport defined next.

7.3 Rotation Bivector [MTW]

☐ The construction of Fermi coordinates may be generalized to observer frames moving along arbitrary worldlines. The characterization of a set of tetrad fields $\{\mathrm{e}_a(\mathbf{x})\}$ as an *accelerated observer frame* is given by its *velocity* and *acceleration* along the path $\mathbf{x}(\lambda)$ of the observer:

7.4

$$\mathbf{u} := \mathrm{e}_0 \quad \dot{\mathbf{u}} := \mathbf{u}\cdot D\mathbf{u} = \frac{D\mathrm{e}_0}{d\lambda} \quad \mathbf{u}\cdot\dot{\mathbf{u}} = 0$$

☐ The motion of the observer frame, relative to parallel transport, is described by the spin connection bivector (4.5) and can be written as a rotation (Lorentz transformation) in spacetime:

7.5

$$\frac{D\mathrm{e}_a}{d\lambda} = \mathbf{u}\cdot D\mathrm{e}_a = \boldsymbol{\Omega}\cdot\mathrm{e}_a \quad \boldsymbol{\Omega} := u^\mu\boldsymbol{\omega}_\mu = \boldsymbol{\omega}_0$$

The *rotation bivector* $\mathbf{\Omega}$ has the Ricci rotation coefficients (4.20) as components:

7.6

$$\mathbf{\Omega} = \frac{1}{2}\omega_{0ab}\,\mathrm{e}^a \wedge \mathrm{e}^b \quad \mathbf{\Omega}\cdot\mathrm{e}_b = \omega_0{}^c{}_b\,\mathrm{e}_c$$
$$\Omega_{ab} = \Omega_{[ab]} := \mathrm{e}_a \cdot \mathbf{\Omega} \cdot \mathrm{e}_b = \omega_{0ab}$$

☐ Being anti-symmetric, the rotation bivector has *six independent components*. This number agrees with the number of components in a Lorentz transformation (three parameters for rotations, plus three parameters for boosts). The components Ω_{0i} may be identified with the *acceleration*, and the spatial components Ω_{ij} with the *frequency of rotation* of the local spatial frame, as measured with respect to the local *non-spinning* observer frame.

7.4 FW Bivector [MTW,DL]

☐ To count as an *observer frame*, an accelerated tetrad must conform to the condition of 'non-rotation' as stated in subsection Observer Tetrad. The spatial basis at each point is still orthonormal, with $\mathrm{e}_i \cdot \mathbf{u} = 0$, but must be transformed, from point to point, in a precise sense by a sequence of infinitesimal Lorentz boosts, *without* additional rotations.

☐ This requirement is *uniquely* met by specifying the rotation vector in the equation of motion (7.5) for the tetrad $\{\mathrm{e}_a\}$, as the so-called *acceleration bivector*, in this particular case called the *Fermi-Walker (FW) bivector*:

7.7

$$\frac{D\mathrm{e}_a}{d\lambda} = \mathbf{\Omega}_{\mathrm{FW}} \cdot \mathrm{e}_a \quad \mathbf{\Omega}_{\mathrm{FW}} := (\dot{\mathbf{u}} \wedge \mathbf{u})$$

The base vectors are then carried along from one instant to the next by a *pure boost* in the $\mathbf{u}$-frame; they are said to be *Fermi-Walker transported*. If the proper 4-acceleration $\dot{\mathbf{u}}(\lambda)$, defined in (7.4), would be zero, the observer would be freely falling and would be parallel transporting his tetrad as in (7.1).

FW bivector

a. respects both the *orthogonality* $\mathbf{u}\cdot\dot{\mathbf{u}} = 0$ and the *normalization* $\mathbf{u}\cdot\mathbf{u} = \mathrm{e}_0\cdot\mathrm{e}_0 = 1$;
b. generates the appropriate Lorentz boosts in the *timelike* plane spanned by the 4-velocity and the 4-acceleration;
c. excludes a rotation in any other plane, in particular, any *spacelike* plane.

☐ Physically, the FW bivector $\mathbf{\Omega}_{\mathrm{FW}}$ represents the acceleration projected into the *instantaneous rest frame* of the observer defined by $\mathbf{u}$. In this frame the projected bivector is strictly *timelike* and the generator of a pure boost.

7.5 FW Transport Equation [MTW]

□ From the FW transport equation (7.7) it is possible, at least in principle, to determine the unit space vectors $\{\mathrm{e}_i(\lambda)\}$ at each point of the curve by solving the differential equations

7.8

$$\frac{D\mathrm{e}_i}{d\lambda} = (\dot{\mathbf{u}} \wedge \mathbf{u}) \cdot \mathrm{e}_i = -\left(\mathrm{e}_i \cdot \frac{D\mathbf{u}}{d\lambda}\right)\mathbf{u}$$

for some orthonormal initial vectors $\mathrm{e}_i(0)$. Equation (7.8) determines the transport of the orthonormal basis $\{\mathbf{u}, \mathrm{e}_i\}$ along an arbitrary timelike curve $\mathbf{x}(\lambda)$, such that $\mathrm{e}_0 = \mathbf{u}$ is always tangent to this curve and the spatial base vectors non-precessing. The initial spatial frame can be arbitrarily chosen on some space-like hypersurface, e.g. the initial data surface. The unique solution obtained this way is called the *Fermi–Walker transported* Lorentz frame along the given curve $\mathbf{x}(\lambda)$.

□ In calculations it is useful to know the connection coefficients. In the FW transport equation, they are only needed on the observer's worldline. Some of the connection coefficients are determined by the FW bivector which has the components $(\dot{\mathbf{u}} \wedge \mathbf{u})_{ab} = 2\dot{u}_{[a}u_{b]}$. Comparing with the rotation bivector (7.6), one finds the Ricci rotation coefficients

7.9

$$\Omega_{ab} = \omega_{0ab} = 2\dot{u}_{[a}u_{b]}$$

□ FW-transport may be *physically* realized in the observer frame by a system of three inertial gyroscopes attached to the spatial directions. The three spacelike tetrad fields should keep their orientation aligned to the system of gyroscopes.

Chapter 8

Schwarzschild Solution

8.1 Schwarzschild Metric [MTW,KL,LS]

□ In January 1916, only a month after publication of Albert Einstein's general theory of relativity, Karl Schwarzschild published an *exact solution* of the vacuum Einstein equations describing the gravitational field outside a spherical mass. Hypothesizing that the metric for this case must be static and spherically symmetric, Schwarzschild obtained the line element that reads, in present day notation:

8.1

$$ds^2 = e^{2\Phi}dt^2 - e^{2\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The scalar parameters Φ, λ are determined by the Einstein equations; see Vacuum Solutions. The metric in the above form was independently arrived at by Johannes Droste, a PhD student of Lorentz, a few months later in 1916.

□ From the Einstein equations, which are differential equations for the metric tensor $g_{\mu\nu}$, Schwarzschild obtained the metric components:

8.2

$$g_{tt} = e^{2\Phi} = 1 - \frac{C}{r} = 1 - \frac{2M}{r} \quad g_{rr} = -e^{2\lambda} = -e^{-2\Phi}$$

The constant C is an integration constant, which is set to $C = 2M$ to be consistent with the *Newtonian gravitational potential* $\Phi = -M/r$ in the weak gravity limit $e^{2\Phi} \simeq 1 + 2\Phi$.

□ The Schwarzschild metric is an *idealized solution* under the assumption that the mass M, creating the gravitational field, is a point at the origin of the coordinates. It is a useful approximation for describing slowly rotating astronomical objects such as stars and planets, including Earth and the Sun. It can also be visualized as all that remains of a star that underwent collapse in the past. But most famously nowadays is the recognition that the Schwarzschild solution is a model for the most basic type of *black hole*.

8.2 Schwarzschild Radius [MTW,LS]

☐ Some important features of the Schwarzschild metric (8.1) are:

a. Far out ($r \to \infty$), the metric component g_{tt} in front of dt^2 becomes one, and the component g_{rr} in front of dr^2 minus one. Thus, in this limit the metric becomes *Minkowskian* in polar coordinates.

$$ds^2 \asymp dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

It implies that asymptotically t and r have their *special-relativistic* significance.

b. Moving inward along decreasing r, one eventually crosses the sphere defined by $r = 2M$. Here the geometry becomes very different: $g_{tt} = 0$ and $g_{rr} = \infty$; moreover, both metric components change sign there. However, *physically* nothing really singular happens at $r = 2M$. As will be discussed in Chapter IX, PG coordinates, the singularity turns out to be an *artifact* of the Schwarzschild coordinates that can be removed by a change of coordinates.

c. Going in all the way, one encounters the only real physical singularity at $r = 0$. Anything falling in would experience infinitely strong gravity and be torn apart there.

☐ The length $r_S := 2M$, in physical units $2GM/c^2$, is named the *Schwarzschild radius*, and the sphere with radius $r = r_S$ the *Schwarzschild horizon*. For almost all astrophysical objects, the Schwarzschild radius is extremely small in comparison to their physical dimensions. For example, the Schwarzschild radius of the Earth is roughly 9 mm, while the Sun, which is 3×10^5 times as massive, has a Schwarzschild radius of approximately 3 km. The ratio becomes large only in close proximity to black holes and other ultra-dense objects such as neutron stars. In fact, a *black hole* is defined as a body whose mass is *entirely contained* within its Schwarzschild radius.

☐ It is worth noting that the Schwarzschild radius is the *only length scale* that appears in the metric. If the time and radial coordinate are rescaled to dimensionless parameters $r/r_S \to r$ and $t/r_S \to t$, the metric (8.1) appears as if $r_S = 1$. Thus, basically all Schwarzschild metrics are the same and for most purposes in studying the geometry of a black hole, one can simple set the Schwarzschild radius equal to one.

8.3 Schwarzschild Coordinates [MTW,LS]

☐ The coordinates $\{\, x^\mu\} := \{t, r, \theta, \phi\}$ of the Schwarzschild metric, or any spherically symmetric metric like (8.1), are simply related to chronometric measurements and observations.

a. The *inclination* from the axis θ, and the *azimuthal angle* ϕ around the axis are the usual polar angles subtended at the origin.

b. The coordinate r is the *radial distance* from the origin. However, in a curved spherically symmetric space, the area of curved successive parallel spheres will not necessarily increase as the square of the distance. The radial Schwarzschild coordinate is defined in such a way that the area of the sphere at radial coordinate r is equal to $4\pi r$, as in flat space, thus, defining the coordinate r as the *area distance.*

☐ The *ruler distance* l, i.e. the distance along the shortest geodesic joining two points obtained from the metric (8.1), is different from r in curved space:

8.3

$$l := \int e^{\lambda} dr \simeq \int_{r_1}^{r_2} \left(1 + \frac{M}{r}\right) dr \simeq r_2 - r_1 + M \log \frac{r_1}{r_2}$$

In the sun's gravitational field, the excess of ruler distance over coordinate distance from the sun's surface to earth is only about 8 km, a discrepancy of one part in 2×10^7 .

☐ In the astronomical praxis other definitions and methods of determining distances are used. These include radar distance, distance from apparent size or luminosity, and distance by parallax, all leading to (slightly) different outcomes.

8.4 Red-shift factor [MTW,LS]

☐ To determine the physical meaning of the time coordinate, one may consider the line element (8.1) for $r, \theta, \phi =$ const defining the *proper time*

8.4

$$d\tau^2 = e^{2\Phi} dt^2$$

Appearing here is the *gravitational time-dilation factor* $\exp -\Phi$ predicted by the equivalence principle.

☐ Physically, this factor may be understood from the photon character of light: the work done by a gravitational field with potential Φ on a particle of gravitational mass $\varepsilon = h\nu$, as it traverses a potential difference, must equal to the gain in the particle's energy. Integrating this difference over a finite path from A to B, one finds the *gravitational red-shift formula*

8.5

$$\frac{\nu_B}{\nu_A} = \sqrt{\frac{g_{tt}(A)}{g_{tt}(B)}} = \frac{e^{-\Phi_B}}{e^{-\Phi_A}} = \left(\frac{1 - 2M/r_A}{1 - 2M/r_B}\right)^{1/2}$$

which is an *exact result* for a stationary metric. It also gives the *gravitational time dilation.* This effect is observable on earth and has, for example, to be taken into account to ensure accurate operation of the Global Positioning System (GPS). A much greater effect is observed in the absorption lines from the surface of white dwarfs.

□ Note that the above argument only determines g_{tt}, just one component of the metric tensor. This metric component, as well as the other components of the metric tensor, were obtained by Schwarzschild by solving the Einstein equations.

8.5 Vierbein and Tetrad [MTW,FK,PDK]

□ In terms of *vierbeins*, the line element of GRT has the general form, see (2.4):

8.6

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta_{ab}e_\mu{}^a e_\nu{}^b dx^\mu dx^\nu$$

The Schwarzschild metric (8.1) is *diagonal* and it is then a simple matter to identify the corresponding vierbeins:

8.7

$$e_\mu{}^a = \mathrm{diag}[e^\Phi, e^\lambda, r, r\sin\theta]$$
$$e^\mu{}_a = \mathrm{diag}[e^{-\Phi}, e^{-\lambda}, r^{-1}, r^{-1}(\sin\theta)^{-1}]$$

□ In general, the *vierbein fields* $\{e_\mu{}^a(x)\}$ define the *local tetrads* $\{\mathbf{e}_a(x)\}$ through (2.6),(2.7). However, in a *diagonal metric*, the coordinate basis $\mathbf{g}_\mu$ will be *orthogonal.* The tetrad base may then be aligned with this coordinate base: $\mathbf{e}_a = \eta_{ab}e_\mu{}^b\nabla x^\mu$, with $\{x^\mu\} := (t, r, \theta, \phi)$ the Schwarzschild coordinates. Normalization to the Minkowski metric gives:

8.8

$$\begin{array}{llll} g_{tt} & := \mathbf{g}_t \cdot \mathbf{g}_t = \mathrm{e}^{2\Phi} & \mathbf{e}_t = e^{-\Phi}\mathbf{g}_t \\ g_{rr} & := \mathbf{g}_r \cdot \mathbf{g}_r = -e^{2\lambda} & \mathbf{e}_r = e^{-\lambda}\mathbf{g}_r \\ g_{\theta\theta} & := \mathbf{g}_\theta \cdot \mathbf{g}_\theta = -r^2 & \mathbf{e}_\theta = r^{-1}\mathbf{g}_\theta \\ g_{\phi\phi} & := \mathbf{g}_\phi \cdot \mathbf{g}_\phi = -r^2\sin^2\theta & \mathbf{e}_\phi = (r\sin\theta)^{-1}\mathbf{g}_\phi \end{array}$$

This choice of tetrads very much simplifies further calculations.

8.6 Connection coefficients [DH,DL,PDK]

□ In the GA formalism, the connection coefficients for the Schwarzschild metric (8.1) can be obtained via the corresponding connection bivectors (4.5). In this particular orthogonal case, the formula to use is (4.14.). Note that from the definition of the gradient $\nabla := \mathbf{g}^\mu\partial_\mu$, it follows that $\nabla t = \mathbf{g}^\mu\partial_\mu t = \mathbf{g}^t$ and $\nabla r = \mathbf{g}^\mu\partial_\mu r = \mathbf{g}^r$. Since the components of the Schwarzschild metric tensor only depend on the radius r :

8.9

$$\nabla := \mathbf{g}^r\partial_r = e^\Phi\mathbf{e}^r\partial_r$$
$$\mathbf{e}_r = e^{-\lambda}\mathbf{g}_r = e^{-\lambda}g_{rr}\mathbf{g}^r = -e^\lambda\mathbf{g}^r$$

□ With the above rules, the connection coefficients for the Schwarzschild solution can directly be calculated from the metric:

8.10

$$\omega_t = -(\partial_r\Phi)e^{\Phi-\lambda}\mathbf{e}_t \wedge \mathbf{e}_r \quad \omega_\theta = e^{-\lambda}\mathbf{e}_r \wedge \mathbf{e}_\theta$$
$$\omega_r = -(\partial_t\lambda)e^{\lambda-\Phi}\mathbf{e}_t \wedge \mathbf{e}_r \quad \omega_\phi = \left(e^{-\lambda}\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta\right) \wedge \mathbf{e}_\phi$$

Since the coefficients in the above connections only depend on the radius, it follows that $e^\lambda\partial_t\lambda = \partial_t e^\lambda = 0$ and

8.11

$$e^\Phi\partial_r\Phi = \frac{1}{\sqrt{1-2M/r}}\frac{M}{r^2} := g$$

☐ Hence, the Schwarzschild connections are determined by *three bivectors* with *three connection coefficients*. It should be noted that the wedges $\wedge$ in (8.10) are actually unnecessary because the base vectors are orthogonal.

8.7 Static Observers [DH,DL,PDK]

☐ The geometric interpretation of the connection coefficients (8.10) may be obtained by analyzing the parallel-transport of the tetrad base defined in (8.8) in the *time direction*. The time-like connection coefficient gives

8.12

$$\mathbf{e}_t \cdot D\mathbf{e}_t = e^{-\Phi}\mathbf{g}_t \cdot D\mathbf{e}_t = e^{-\Phi}\omega_t \cdot \mathbf{e}_t$$
$$= -e^{-\Phi}\frac{M}{r^2}\mathbf{e}_t \wedge \mathbf{e}_r \cdot \mathbf{e}_t = g\mathbf{e}_r$$

☐ The *gravitational acceleration* g is as defined in (8.11). It matches with g being the gradient of the relativistic potential measured with respect to the *radial ruler distance* (8.3):

8.13

$$g = |\nabla\Phi| := \frac{d\Phi}{dl} = \frac{d\Phi}{dr}\frac{dr}{dl} = \frac{M/r^2}{\sqrt{1-2M/r^2}}$$

This is the Newtonian value multiplied by the *gravitational dilatation factor*, which becomes increasingly large as the Schwarzschild horizon is approached.

☐ The acceleration vector $g\mathbf{e}_r$ in (8.12) is radially outward pointing since the static frame $\{\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ needs to accelerate away from the central object to avoid falling in. The thrust required to maintain position is given by the magnitude of the acceleration vector (8.13). This is a constant, since the position is fixed for these observers. It is only possible to remain at rest outside the horizon.

☐ The parallel transport of the *spatial vectors* gives

8.14

$$\mathbf{e}_t \cdot D\mathbf{e}_r = e^{-\Phi}\mathbf{g}_t \cdot D\mathbf{e}_r = g\mathbf{e}_r \wedge \mathbf{e}_t \cdot \mathbf{e}_r$$
$$\mathbf{e}_t \cdot D\mathbf{e}_\theta = 0 \quad \mathbf{e}_t \cdot D\mathbf{e}_\phi = 0$$

With the notation $\mathbf{u} := \mathbf{e}_t = e^{-\Phi}\mathbf{g}_t$ and $\dot{\mathbf{u}} := g\mathbf{e}_r = (M/r^2)\mathbf{g}_r$, equations (8.12,14) are recognized as an instance of the FW-transport equation (7.7). So, the static frames $\{\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ may be identified as FW-transported *non-spinning observer tetrads*.

Chapter 9

Spherically Symmetric Spacetimes

9.1 General Line Element [DH,DL,FK]

□ The Schwarzschild solution may give the simplest form of the metric, but it does not offer the simplest or most physically transparent interpretation. For one, it is ill defined at the Schwarzschild radius because the coordinate system is singular there. However, for the case of *stationary, spherically-symmetric* spacetimes, there are exact solutions of the Einstein equations other than Schwarzschild's.

□ It is relevant to note here that the *stationarity* condition is not the same as the *static* condition, which is the statement that $\mathbf{g}_t \cdot \mathbf{g}_r = 0$. The static condition is highly restrictive, and leads automatically to the Schwarzschild metric. Stationarity, on the other hand, amounts to independence of all quantities of the time variable t.

□ By an analysis of Einstein's vacuum (source-free) equations, it has been shown that the vierbeins associated with stationary, *spherically-symmetric spacetimes*, must be of the general form

9.1

$$[e_\mu{}^a] = \begin{bmatrix} g_1 & -g_2 & 0 & 0 \\ -f_2 & f_1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{bmatrix} \qquad [e^\mu{}_a] = \begin{bmatrix} f_1 & f_2 & 0 & 0 \\ g_2 & g_1 & 0 & 0 \\ 0 & 0 & r^{-1} & 0 \\ 0 & 0 & 0 & (r\sin\theta)^{-1} \end{bmatrix}$$

where f, g are functions of the radius r only; the determinant of the non-diagonal bloc has the value $f_1 g_1 - f_2 g_2 = 1$; this last condition guarantees that the frames defined by (9.1) satisfy the *torsion-free* condition (1.26).

□ By substitution of (9.1) in (8.6) one obtains the general *spherically symmetric* line element

9.2

$$ds^2 := \mathbf{g}_\mu \cdot \mathbf{g}_\nu dx^\mu dx^\nu =$$
$$(g_1^2 - g_2^2)dt^2 + 2(f_1 g_2 - f_2 g_1)dtdr - (f_1^2 - f_2^2)dr^2 - r^2 d\Omega^2$$

Here $d\Omega^2 := (d\theta^2 + \sin^2\theta d\phi^2)$ is an abbreviation for the metric of the unit sphere.

9.2 Connection Bivectors [DH,DL,FK]

☐ The calculation of the connection bivectors starts with the identification of metric tensor component from (9.2) and the relation between the coordinate basis and the tetrad base from (9.1):

9.3

$$\begin{array}{rll} g_{tt} = & \mathbf{g}_t \cdot \mathbf{g}_t = \mathrm{g}_1^2 - \mathrm{g}_2^2 & \mathbf{g}_t = g_1 \mathbf{e}_t - g_2 \mathbf{e}_r \\ g_{rr} = & \mathbf{g}_r \cdot \mathbf{g}_r = f_2^2 - f_1^2 & \mathbf{g}_r = f_1 \mathbf{e}_r - f_2 \mathbf{e}_t \\ g_{\theta\theta} = & \mathbf{g}_\theta \cdot \mathbf{g}_\theta = -r^2 & \mathbf{g}_\theta = r \mathbf{e}_\theta \\ g_{\phi\phi} = & \mathbf{g}_\phi \cdot \mathbf{g}_\phi = -r^2 \sin^2\theta & \mathbf{g}_\phi = r \sin\theta \mathbf{e}_\phi \\ g_{tr} = & \mathbf{g}_t \cdot \mathbf{g}_r = f_1 g_2 - g_1 f_2 & \end{array}$$

These relations imply that the tetrad base $\{\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ is *orthonormal.*

☐ The *connection coefficients* may then be obtained by way of the Snygg formula (4.14):

9.4

$$\begin{aligned} \omega_t &= -(g_1 \partial_r g_1 - g_2 \partial_r g_2) \mathbf{e}_t \wedge \mathbf{e}_r \\ \omega_r &= -(f_1 \partial_r g_2 - f_2 \partial_r g_1) \mathbf{e}_t \wedge \mathbf{e}_r \\ \omega_\theta &= g_1 \mathbf{e}_r \wedge \mathbf{e}_\theta - g_2 \mathbf{e}_t \wedge \mathbf{e}_\theta \\ \omega_\phi &= \sin\theta \, (g_1 \mathbf{e}_r - g_2 \mathbf{e}_t) \wedge \mathbf{e}_\phi + \cos\theta \mathbf{e}_\theta \wedge \mathbf{e}_\phi \end{aligned}$$

No derivatives of f_1, f_2 appear.

☐ A comparison of (9.2) with (8.1) shows that the *Schwarzschild metric* corresponds to the parameter choice

9.5

$$f_1 = \frac{1}{g_1} \quad f_2 = 0 \quad g_1^2 = 1 - \frac{2M}{r} = e^{2\Phi} \quad g_2 = 0$$

Substituting these parameters values into (9.4), one recovers the Schwarzschild connection coefficients (8.10).

☐ Note that $f_1 g_2 - f_2 g_1 = 1$ in this case. However, there are other possible choices; for example, $f_1^2 - f_2^2 = 0$ leads to the Eddington-Finkelstein metric, and $f_1^2 - f_2^2 = 0$, to the *Painlevé-Gullstrand metric.*

9.3 PG Coordinates [DL,FK]

☐ The PG-metric, independently found by Paul Painlevé and Allvar Gullstrand in 1921 by solving the vacuum Einstein equations, is given by a line element with a *non-diagonal term*

9.6

$$ds^2 = dt^2 - \left(dr + \sqrt{2M/r}dt\right)^2 - r^2 d\Omega^2$$

This expression obtains from the general metric (9.2) with the parameter choice

9.7

$$\begin{array}{llll} f_1 = 1 & f_2 = 0 & \rightarrow & f_1^2 - f_2^2 = 1 \\ g_1 = 1 & g_2 = -\sqrt{2M/r} & \rightarrow & g_1^2 - g_2^2 = e^{2\Phi} \end{array}$$

The PG-metric may be understood as a *coordinate transformation* of the Schwarzschild metric as shown by Georges Lemaître in 1933. [Wikipedia: Gullstrand–Painlevé coordinates].

☐ A striking property of the PG-metric (9.6) is that *spacelike* surfaces $t =$ const , $dt = 0$, are *intrinsically flat*, i.e. they have the metric of a Euclidean three-dimensional space in spherical polar coordinates:

9.8

$$-ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Moreover, these space-like hypersurfaces are *orthogonal* to the time direction:

9.9

$$\mathbf{e}_t \cdot \mathbf{e}_r = g_{tr} - \sqrt{\frac{2M}{r}} g_{rr} = 0$$

The information about the spacetime curvature is entirely encoded in the off-diagonal component of the metric tensor.

☐ The PG-metric has advantages over the Schwarzschild metric. For one, in PG-coordinates the Schwarzschild metric is *regular* on the whole domain $r > 0$. So, it reveals that the Schwarzschild radius is a mere coordinate singularity that can be removed by a new choice of coordinates. Only the origin of the spherical coordinates is a real singularity. Moreover, the PG-coordinates are key to a simple physical picture of black holes and the gravitational collapse of stars. Close to the central mass, the spacetime (9.6) describes a *black hole* with an event horizon at the Schwarzschild radius, while at large distance it reproduces the results of *Newtonian gravity*.

9.4 PG Observers [DL,FK]

☐ The substitution of the parameters (9.7) into (9.4) gives the *PG connection bivectors*

9.10

$$\omega_t = -\frac{M}{r^2}\mathbf{e}_t \wedge \mathbf{e}_r \qquad \omega_\theta = \left(\mathbf{e}_r + \sqrt{\frac{2M}{r}}\mathbf{e}_t\right) \wedge \mathbf{e}_\theta$$
$$\omega_r = \sqrt{\frac{M}{2r^3}}\mathbf{e}_r \wedge \mathbf{e}_t \qquad \omega_\phi = \sin\theta \left(\mathbf{e}_r + \sqrt{\frac{2M}{r}}\mathbf{e}_t\right) \wedge \mathbf{e}_\phi + \cos\theta \mathbf{e}_\theta \wedge \mathbf{e}_\phi$$

☐ A *PG-observer* is an observer tied to the tetrad $\{\mathbf{e}_t, \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$. From (9.3), with the parameter values (9.7), the *observer velocity* in the coordinate basis is seen to be

9.11

$$\mathbf{u} := \mathbf{e}_t = \mathbf{g}_t - \mathbf{g}_r\sqrt{2M/r}$$

The observer velocity is then found to satisfy the *geodesic equation*

9.12

$$\mathbf{u} \cdot D\mathbf{u} = \left(\omega_t - \omega_r\sqrt{2M/r}\right) \cdot \mathbf{u} = 0$$

Thus, the PG observer frame is *free-falling* and *non-spinning* because also the spatial frame is parallel transported along the worldline of the observer: $\mathbf{e}_t \cdot D\mathbf{e}_j = 0$, $j = r, \theta, \phi$.

☐ In PG-coordinates $\{\ x^\mu\} := \{t, r, \theta, \phi\}$, the velocity (9.11), affinely parametrized by τ, has the components

9.13

$$u^\mu = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}) = \left(1, -\sqrt{2M/r}, 0, 0\right)$$

with respect to the coordinate base; notation $\dot{t} := dt/d\tau$ etc. Thus, the free-falling observer goes radially inward with 3-velocity $\dot{r} = \sqrt{2M/r}$, which equals the negative of the *Newtonian escape velocity.* At the event horizon, the observer reaches the velocity of light $c = 1$. There is no discontinuity or singularity at the event horizon.

☐ The value of the time component $\dot{t} = dt/d\tau = 1$ implies that the PG time-coordinate t is the *proper time* of the geodesic observer. The GP-coordinates could have been defined by this requirement.

9.5 Schwarzschild Black Hole [DL,LS]

☐ Photons with frequency 4-vector $\mathbf{k} = (\nu, \vec{k})$, $\nu = \left|\vec{k}\right|$, follow *null trajectories* $\mathbf{k}^2 = 0$ specified by the geodesic equation $\mathbf{k} \cdot D\mathbf{k} = 0$. A radial null trajectory has tangent

9.14

$$\mathbf{k} = \nu(\mathbf{e}_t \mp \mathbf{e}_r) \qquad \nu := \mathbf{k} \cdot \mathbf{e}_t$$

The minus sign applies to a light ray *towards* the center; the plus sign to an *outward* light ray. The frequency ν is measured by a radially free-falling observer (at rest at infinity).

□ From (9.3) with (9.7), one derives the tangent in coordinate representation

9.15

$$\mathbf{k} = \nu \mathbf{g}_t - \nu \left(\pm 1 + \sqrt{2M/r} \right) \mathbf{g}_r$$

It follows that the velocity along the photon path is

9.16

$$\frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}}$$

This result could have been directly obtained from the *PG-metric* (9.6.) by setting $ds = 0$, $d\theta = 0$, $d\phi = 0$.

□ At the *event horizon*, the speed of light (9.16) vanishes for rays shining outward (+): $dr/dt = 0$. More strangely, inside the event horizon, $r < r_{\mathrm{S}}$, this light ray moves *toward* the center, implying that light cannot escape. It gets stuck at the event horizon which marks the boundary between the interior region that cannot signal to the outer region. Thus, everything inside the event horizon is hidden from the outside world: it is a *Black Hole*.

□ Matter can only move *inwards* from the event horizon. If any object collapses to within its event horizon, it must carry on collapsing to form a central singularity. There is no force capable of preventing the collapse. This is because matter is always constrained to follow timelike paths, and if the entire future light-cone points inwards towards the singularity, no matter can escape. All paths for in-falling matter terminate on the singularity.

Chapter 10

Curvature

10.1 Riemann Tensor [MTW,SC,PDK]

□ One of the characteristics of a flat manifold is that two coderivatives $D_\mu D_\nu$ commute; in curved space they do not:

10.1

$$[D_\mu, D_\nu]\,\mathbf{g}_\kappa := \mathbf{R}_{\mu\nu} \cdot \mathbf{g}_\kappa$$

This defines the *curvature bivector* $\mathbf{R}_{\mu\nu}$ as a measure for the noncommutativity of the coderivative. Geometrically, the bivector calculates the difference in a vector that is parallel transported around a loop. If and only if the curvature is zero everywhere, space is flat. Physically, the curvature bivector fully encodes the gravitational interaction.

□ The curvature bivector may be expanded with respect to a coordinate or tetrad basis

10.2

$$\mathbf{R}_{\mu\nu} = \frac{1}{2} R_{\mu\nu\kappa\lambda}\mathbf{g}^\kappa \wedge \mathbf{g}^\lambda = \frac{1}{2} R_{\mu\nu ab}\mathbf{e}^a \wedge \mathbf{e}^b$$

The mixed components tensor is the one defined in (6.3). The coefficients $R_{\mu\nu\kappa\lambda}$ are the components of the *Riemann curvature tensor* (Riemann for short) as it is usually presented in the tensorial formalism of GRT. By explicitly evaluating the commutator (10.1), one derives an expression for the Riemann in terms of the Levi-Civita connection coefficients (1.15):

10.3

$$R^\mu{}_{\nu\kappa\lambda} = 2\partial_{[\kappa}\Gamma^\mu_{\ \lambda]\nu} + 2\Gamma^\mu_{\ [\kappa|\rho}\Gamma^\rho_{\ \lambda]\nu}$$

This is a complicated expression of Γ's that involves first and second order derivatives of the metric tensor, which makes explicit calculation a daunting task. However, there are a number of symmetries and constraints that greatly reduce the number of independent components.

☐ The components of the Riemann are *anti-symmetric* in both their first and second pairs of indices. One can also interchange the two pairs of indices:

10.4

$$\begin{array}{llll} R_{\mu\nu\kappa\lambda} = R_{[\mu\nu]\kappa\lambda} & \text{(a)} & R_{\mu\nu\kappa\lambda} = R_{\mu\nu[\kappa\lambda]} & \text{(b)} \\ R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu} & \text{(c)} & R_{\mu[\nu\kappa\lambda]} = 0 & \text{(d)} \end{array}$$

The anti-symmetry propriety (a) is obvious from definition (10.1) and the anti-symmetry (b) can be read off from (10.2) or (10.3). Propriety (c) may be proven by considering a local inertial frame such that the Γ's in the last term of equation (10.3) vanish. From this simplified expression, with (1.29) inserted, one can prove symmetry relations (c) and (d). The latter only holds in spaces *without* torsion. Together, these symmetries reduce the *number of independent components* of the Riemann to 20.

10.2 Second Cartan Equation [DL,PDK]

☐ For the purpose of relating the curvature tensor to the spin (Lorentz) connection, it is useful to establish a link with the Cartan formalism. In this formalism the curvature of a manifold is characterized by the tetrad curvature 2-form

10.5

$$\mathbf{R}^{ab} := e_\mu{}^a e_\nu{}^b \mathbf{R}^{\mu\nu} = \frac{1}{2} R^{ab}_{\mu\nu} \mathbf{g}^\mu \wedge \mathbf{g}^\nu = \frac{1}{2} R^{ab}_{cd} \mathbf{e}^c \wedge \mathbf{e}^d$$

Inserting the tensor components as specified in (6.3) and subsequently writing out the right-hand side in terms of the connection 1-form defined in (4.22), one obtains the *second Cartan structure equation* [Wikipedia: Spin Connection]:

10.6

$$\mathbf{R}^a{}_b = \mathrm{d}\boldsymbol{\omega}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \boldsymbol{\omega}^c{}_b$$

☐ The equivalent GA-representation is obtained by substituting (6.3) into the last member of equation (10.2). The result is

10.7

$$\mathbf{R}_{\mu\nu} = \partial_\mu \boldsymbol{\omega}_\nu - \partial_\nu \boldsymbol{\omega}_\mu + [\boldsymbol{\omega}_\mu, \boldsymbol{\omega}_\nu]$$

defining the curvature bivector in terms of the *spin connection bivector* (4.5). The derivation shows that the GA curvature bivector (10.7) is completely *isomorphic* to the Cartan curvature 2-form (10.6), offering an alternative way to describe and calculate curvature.

☐ The curvature bivector (10.7) may be thought of as a point-wise *multi-linear map*

10.8

$$\mathbf{R}_{\mu\nu} := \mathrm{Riem}(\mathbf{g}_\mu \wedge \mathbf{g}_\nu)$$

from tangent bivectors to tangent bivectors; i.e. $\mathrm{Riem}(\mathbf{g}_\mu \wedge \mathbf{g}_\nu)$ is a bivector-valued function of bivectors. In the mixed indices representation at the-right-hand side of (10.2), it connects a bivector area $\mathbf{g}_\mu \wedge \mathbf{g}_\nu$ with a bivector $\mathbf{e}^a \wedge \mathbf{e}^b$ which is the generator of the rotation that a vector experiences when transported around the area perimeter.

10.3 First Bianchi Identity [SC,DR,PDK]

☐ The Riemann (10.2) satisfies the *first Bianchi identity*, also called the 'algebraic' Bianchi identity. It may be obtained by taking the exterior coderivative of the first Cartan structure equation (4.26)

10.9

$$\mathrm{D}\mathbf{T}^a = \mathrm{d}\mathbf{T}^a + \boldsymbol{\omega}^a{}_b \wedge \mathbf{T}^b = \mathbf{R}^a{}_b \wedge \mathbf{e}^b$$

where $\mathbf{T}^a$ is the torsion form as it appears in the first Cartan equation; $\boldsymbol{\omega}^a{}_b$ is the Cartan connection (4.22). The right-hand side is identically equal to the left-hand side, as required of an identity. Proof: just substitute the first Cartan equation into (10.9) and then use the second Cartan equation (10.6). [Wikipedia: Spin Connection]

☐ If spacetime is torsionless, identity (10.9) reduces to

10.10

$$\mathbf{R}_{\mu\nu} \wedge \mathbf{g}^\nu = \frac{1}{2} R_{\mu\nu\kappa\lambda} \mathbf{g}^\kappa \wedge \mathbf{g}^\lambda \wedge \mathbf{g}^\nu = 0$$

which is equivalent to the vanishing of the antisymmetric part of the Riemann as in (10.4d). Although known as the 'first Bianchi identity', this property was actually discovered by Gregorio Ricci-Curbastro and Tullio Levi–Civita; it is not so much an identity as a theorem, because it is only valid in torsionless spacetimes.

☐ The first Bianchi identity may also be derived directly from definition (10.1) of the curvature bivector. By using the property $D_\mu \mathbf{g}_v - D_\nu \mathbf{g}_\mu = 0$ of a torsionless spacetime, equation (1.25), one may manipulate the left-hand side of (10.1) into

10.11

$$[D_\mu, D_\kappa]\mathbf{g}_\nu - [D_\nu, D_\kappa]\mathbf{g}_\mu = \mathbf{R}_{\mu\nu} \cdot \mathbf{g}_\kappa$$

With the notation (10.8), this may be translated into the *cyclic form* of the first Bianchi identity:

10.12

$$\mathrm{Riem}(\mathbf{g}_\mu \wedge \mathbf{g}_\nu) \cdot \mathbf{g}_\kappa + \mathrm{cyclic} = 0$$

valid for spacetimes without torsion.

10.4 Second Bianchi Identity [SC,JCS]

☐ The *second Bianchi identity* is a differential identity obtained by taking the exterior coderivative of the curvature 2-form:

10.13

$$\mathrm{D}\mathbf{R}^a{}_b = \mathrm{d}\mathbf{R}^a{}_b + \boldsymbol{\omega}^a{}_c \wedge \mathbf{R}^c{}_b - \boldsymbol{\omega}^c{}_b \wedge \mathbf{R}^a{}_c = 0$$

<u>Proof</u>: Substitute the second Cartan equation (10.6) and use the property $\mathrm{dd}\mathbf{A} = 0$ for general $\mathbf{A}$; see definition (1.7). After some algebra, taking into account the (anti-) commutation rule $(-1)^p$, one obtains (10.13). [Wikipedia: Spin Connection]

☐ For the case of *no torsion*, the second Bianchi identity may be written in the component form

10.14

$$D_\rho R_{\mu\nu\kappa\lambda} + D_\mu R_{\nu\rho\kappa\lambda} + D_\nu R_{\rho\mu\kappa\lambda} = 3D_{[\rho} R_{\mu\nu]\kappa\lambda} = 0$$

This equation may be also proven on the basis of the *Jacobi identity*, since it basically expresses

10.15

$$[[D_{[\mu}, D_\nu], D_{\kappa]}] = 0$$

☐ By operating with the Jacobi operator on an arbitrary vector and using definitions (10.1) and (10.8), one derives the second Bianchi identity in the *cyclic form*

10.16

$$D_\rho \, \mathrm{Riem}(\mathbf{g}_\mu \wedge \mathbf{g}_\nu) + \mathrm{cyclic} = 0$$

☐ The above version of the second Bianchi identity is true *if and only if* the connection is symmetric and the torsion is zero.

10.5 Ricci Tensor [MTW,PDK]

☐ From the Riemann (10.2), which contains all information about curvature, one defines other tensors important in GRT, with less detailed curvature information. First of all the *Ricci tensor* (Ricci for short), either in the coordinate or tetrad basis, is obtained by the contraction of the first and third indices of the Riemann:

10.17

$$R_{\mu\nu} := g^{\kappa\lambda} R_{\kappa\mu\lambda\nu} \qquad R_{ab} := \eta^{cd} R_{cadb}$$

The Ricci is *symmetric* on account of the interchange and skew symmetries (10.4a,b,c). Note that the alternative contraction over indices 1 and 4 differs by a minus sign. It is a matter of choice what convention to use, because there is no generally accepted convention for the sign of either the Riemann tensor or the Ricci tensor.

☐ By a further contraction of the Ricci with the metric one obtains the *Ricci scalar* (also called scalar curvature), the simplest measure of curvature:

10.18

$$R := g^{\mu\nu} R_{\mu\nu} = \eta^{ab} R_{ab}$$

Both the Ricci and the Ricci scalar have the dimension of an *inverse squared length.* This means that the square root of the inverse of the curvature gives a typical length scale that one may use to estimate gravitational effects.

☐ In the GA-formalism, the Ricci is defined from equation (10.2) by the contraction

10.19

$$\mathbf{R}_\mu := \mathbf{g}^\nu \cdot \mathbf{R}_{\nu\mu} = \mathbf{g}^\nu \cdot \mathrm{Riem}(\mathbf{g}_\nu \wedge \mathbf{g}_\mu) := \mathrm{Ric}(\mathbf{g}_\mu)$$

Thus, in GA, the Ricci is a *linear operator* on the tangent space that maps vectors to vectors, consistent with the tensor definitions (10.17):

10.20

$$\mathrm{Ric}(\mathbf{g}_\mu) = R_{\mu\nu}\mathbf{g}^\nu \qquad \mathrm{Ric}(\mathbf{e}_a) = R_{ab}\mathbf{e}^b$$

Similarly, consistent with (10.18), the Ricci scalar

10.21

$$R = \mathbf{g}^\mu \cdot \mathbf{R}_\mu = (\mathbf{g}^\mu \wedge \mathbf{g}^\nu) \cdot \mathbf{R}_{\nu\mu} = g^{\mu\nu} R_{\mu\nu}$$

is obtained by contracting the Ricci, or directly the Riemann.

10.6 Einstein Tensor [SW,MTW]

☐ An especially useful form of the second Bianchi identity comes from contracting twice on (10.14) to obtain the so-called *contracted Bianchi identity* (first derived by the mathematician Aural Voss, 1880):

10.22

$$2D_\mu R^\mu{}_\nu - D_\nu R = 0$$

where $R_{\mu\nu}$ is the Ricci (10.17) and R the Ricci scalar (10.18).

☐ By defining the particular combination, known as the *Einstein tensor*

10.23

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \qquad \mathrm{G}(\mathbf{g}_\mu) := \mathrm{Ric}(\mathbf{g}_\mu) - \frac{1}{2} R \mathbf{g}_\mu$$

it is simply seen that the contracted Bianchi identity (10.22) is equivalent to

10.24

$$D_\mu G^\mu{}_\nu = D_\mu G^{\mu\nu} = 0 \qquad D_\mu \mathrm{G}(\mathbf{g}^\mu) = 0$$

That is, the Einstein tensor is *covariantly conserved* on account of the contracted Bianchi identity (10.22) that holds for any *torsionless* spacetime.

☐ *Unique* properties of the Einstein tensor are:

a. identically zero when spacetime is *flat*;
b. *symmetric* due to the symmetry of the Ricci tensor and the metric;
c. *linearly* related to the Riemann curvature tensor;
d. constructed from the *Riemann* and the *metric*;
e. determined by *first* and *second* derivatives of the metric only;
f. *covariantly* conserved.

In $d = 4$ dimensions, the most general symmetric, divergence-free tensor of rank two constructed from the metric and its first two derivatives is $aG_{\mu\nu} + bg_{\mu\nu}$ with constants a and b. [Wikipedia: Lovelock's Theorem]

Chapter 11

Field Equations

11.1 Einstein Equation [MTW,KL]

□ In the Newton theory of gravitation, mass is the source of the gravitational force. This suggests for GRT that mass, or rather *energy* since it is a conserved quantity, is affecting geometry. However, in relativity theory, energy is only conserved in combination with momentum. Therefore Einstein surmised that the *energy-momentum tensor* $\mathbf{t} = \mathrm{t}(\mathbf{g}_\mu)$, with components $t^{\mu\nu}$, is the frame-independent geometric object that must act as the source of gravity.

□ It is a theorem of relativistic field theories that the energy-momentum tensor is *symmetric*: $t^{\mu\nu} = t^{\nu\mu}$; this makes index raising/lowering unambiguous. Moreover, to uphold the law of conservation of energy-momentum, this tensor must be covariantly conserved

11.1

$$D_\mu t^{\mu\nu} = 0 \qquad D_\mu\, \mathrm{t}(\mathbf{g}^\mu) = 0$$

□ In the now famous article of November 25, 1915, Albert Einstein derived his *gravitational field equation* by setting the Einstein tensor (10.23) proportional to the energy-momentum tensor:

11.2

$$G_{\mu\nu} = \kappa t_{\mu\nu} \qquad \mathrm{G}(\mathbf{g}_\mu) = \kappa\, \mathrm{t}(\mathbf{g}_\mu)$$

The left-hand side describes the geometry of space-time; the right-hand side the matter content of the universe. This simple looking equation actually stands for a complicated set of *second order non-linear differential equations* that relate the local spacetime curvature, i.e. the metric tensor, to the local density and flow of energy and momentum, quantified by the energy-momentum tensor. Hence, one usually speaks of Einstein equations (plural).

□ The Einstein *gravitational constant* κ is a proportionality factor that is determined by a comparison with the Newton theory of gravitation in the appropriate limit. This yields

$\kappa = 8\pi$, or in physical units $\kappa = 8\pi G/c^4$, with Newtons gravitational constant and the speed of light, made explicit. This constant is extremely small: $\kappa \approx 2 \times 10^{-43}$ N^{-1}. Only in the presence of matter under extreme conditions, the effect of gravity on space and time can be strong.

11.2 Vacuum Solutions [SW,MTW]

☐ The Einstein field equations can have non-trivial solutions even in regions where there are *no sources*, i.e. in regions of spacetime that are devoid of matter and radiation. There the Einstein tensor must vanish:

11.3

$$\mathrm{t}(\mathbf{g}_\mu) = 0 \to \mathrm{G}(\mathbf{g}_\mu) = 0$$

This implies that the Ricci tensor vanishes: $\mathrm{Ric}(\mathbf{g}_\mu) = 0 \to R = 0$. But the relationship between the Ricci and Riemann is such that the vanishing of the Ricci not necessarily implies that the Riemann should be zero. If the Riemann tensor is *not zero*, then spacetime must be curved. Such solutions satisfying the Einstein equations (11.3) are called *vacuum solutions*. One of those is the *Schwarzschild solution* (8.1), which is the first, and arguably the most important, non-trivial vacuum solution of the Einstein field equations; see Chapter VIII.

☐ In 1917 Einstein proposed to add to his tensor a *new free constant* Λ according to

11.4

$$G_{\mu\nu} \to G_{\mu\nu} - \Lambda g_{\mu\nu} \qquad \mathrm{G}(\mathbf{g}_\mu) \to \mathrm{G}(\mathbf{g}_\mu) - \Lambda \mathbf{g}_\mu$$

thereby relaxing the requirement that $G_{\mu\nu}$ vanishes when spacetime is flat. The new tensor would still have all other properties of the original, but it would allow a *static* unchanging universe as one particular solution of his equations. Then, in 1929, Edward Hubble discovered that the universe was expanding, as Aleksandr Friedmann and Georges Lemaitre had theoretically predicted a few years earlier. Subsequently, Einstein retracted his proposal calling it “the biggest blunder of my life”.

☐ Still, to this day, the constant Λ, now called the *cosmological constant*, is a parameter in cosmological models to explain the actually observed accelerating expansion of the universe. It is also the simplest explanation for the existence of dark energy in the universe. The cosmological constant Λ is a dimensionful parameter with units of inverse length squared. In the Einstein equation it has the same effect as an intrinsic energy density of the vacuum - the lowest energy state of the universe. This is now the common interpretation of the cosmological constant in the *standard model of cosmology* known as the ΛCDM model. [Wikipedia: Λ-CDM model]

11.3 Einstein-Hilbert Action [MTW,JY]

☐ The *Einstein–Hilbert action* in general relativity is the action that yields the Einstein field equations through the stationary-action principle. This allows comparison of general relativity with other classical field theories formulated in terms of an action (such as electromagnetism). Most importantly, matter can be coupled to gravity by adding to the Einstein–Hilbert action a matter action describing any matter fields appearing in the theory; see e.g. ECKS Theory.

☐ By general covariance, an action for the metric $g_{\mu\nu}$ will have to take the form

11.5

$$S[g] := \int d^4x \sqrt{|g|} R[g_{\mu\nu}]$$

Here, $d^4x\sqrt{|g|}$ is the invariant volume measure and R a scalar constructed from the metric. The simplest choice is the *Ricci scalar* (10.18). It is also the unique choice if one is looking for a scalar constructed only from first and second derivatives of the metric. This simple and elegant action, known as the *Einstein-Hilbert action*, was proposed by Hilbert a few days before Einstein presented his final form of the gravitational field equations on November 25, 1915.

☐ To give the Einstein-Hilbert action the dimension of an action a pre-factor is required, conventionally written as $1/2\kappa$, where $\kappa = 8\pi G/c^{-4}$ is the *Einstein gravitational constant*. One may also include a cosmological constant Λ by the substitution $R \to R + 2\Lambda$:

11.6

$$S_{\text{EH}}[g] = \frac{1}{2\kappa} \int d^4x \sqrt{|g|} \left(R[g_{\mu\nu}] + 2\Lambda\right)$$

☐ Since in GRT the Levi-Civita connection can be expressed in terms of the metric, see (1.29), the Einstein-Hilbert action is taken to be a functional of the (inverse) metric tensor. It can be shown that the Euler-Lagrange equations following upon variation of (11.6) indeed lead to the correctly normalized Einstein equations. [Wikipedia: Einstein–Hilbert action]

11.4 Einstein-Palatini Action [DP]

☐ The canonical Levi-Civita connection is characterized by the fact that the connection is compatible with the metric and has no torsion. However, a priori, *metric* and *connection* are separate concepts, and the notion of curvature can be defined for arbitrary connections. To take metric and connection as *independent variables* in the action principle, might have been suggested by work of Palatini (1919). Fact is that the implementation of this idea was first and independently explored by Einstein in his search for unification of gravity and electrodynamics (1922-32).

☐ In the Einstein-Palatini framework, as it is now commonly known, the Einstein-Hilbert action integral (11.6) is written in terms of the Ricci tensor (10.17) defined by contraction of the Riemann (10.3), so that the Ricci tensor depends on the connection but *not explicitly* on the metric:

11.7

$$S_{\mathrm{EP}}[g,\Gamma] := \frac{1}{2\kappa}\int d^4x\sqrt{|g|}\,(g_{\mu\nu}R^{\mu\nu}[\Gamma] + 2\Lambda)$$

The pair of *metric* and *connection* $\left\{g_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu}\right\}$ constitute the *basic gravitational variables*, without the restriction that the connection is torsion free, i.e. symmetric in its lower indices. The Einstein equations are then obtained by varying the (inverse) metric, and the metric compatible and torsion-free Levi-Civita connection by varying the connection. It is called a *first-order formulation* as the variables to vary over involve only up to first derivatives in the action.

☐ In the actual variational calculation it is convenient to introduce the *contortion tensor* $C^{\lambda}_{\mu\nu}$, i.e. the deviation from the Levi-Civita connection, as the variable to be varied. Solving the Euler-Lagrange equations of motion, one is led to the conclusion that the connection is the Levi-Civita one, up to an *arbitrary* vector A_μ:

11.8

$$\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu}\Big|_{\mathrm{LC}} + A_\mu\delta^{\lambda}_{\nu} \qquad A_\mu := C^{\sigma}_{\sigma\mu}$$

Since A_μ can have any value, one may set this vector equal to zero. In fact, it was already stated by Einstein in 1925 that, in order to obtain the correct equations of motion, the *trace of the torsion tensor* (1.23) must vanish: $T^{\nu}_{\mu\nu} = 3A_\mu = 0$.

☐ However, there is altogether no need to impose any condition because the arbitrariness of A_μ corresponds to a (Noether) *gauge symmetry* of the action, as is easily verified. As a consequence, the equations of motion are invariant. Therefore, one may conclude that the theory described by the Einstein-Palatini action (11.7) is indeed Einstein's GRT, without any ambiguity.

11.5 Vierbein E-P Action [DP]

☐ Alternatively, the Einstein-Palatini action (11.7) may be formulated in terms of *vierbein-* and *spin-connection* fields $\{e^{\mu}{}_{a}, \Sigma_{\mu}{}^{a}{}_{b}\}$ as basic dynamical variables. From their definitions (2.6,7) and (4.4), it is seen that they are 1-form fields on $\mathcal{M}_4$ taking values, respectively, in V_4 and in the Lie algebra of the Lorentz group $\mathrm{SO}(1,3)$.

☐ By identifying the Ricci scalar as the double contraction of the 2-form *gauge curvature tensor* (6.3) induced by the spin connection:

11.9

$$R = \eta^{ab}\eta^{cd}R_{cadb} = R^{ab}_{ab} := e^{\mu}{}_{a}e^{\nu}{}_{b}R^{ab}_{\mu\nu}[\Sigma]$$

the action gets the *vierbein Einstein-Palatini* form

11.10
$$S_{\mathrm{EP}}[e,\Sigma] = \frac{1}{2\kappa}\int d^4x\,\mathrm{e}\left(e^{\mu}{}_{a}e^{\nu}{}_{b}R^{ab}_{\mu\nu}[\Sigma] + 2\Lambda\right)$$

with $\mathrm{e} := \det[\mathrm{e}_{\mu}{}^{\mathrm{a}}] = \sqrt{|\mathrm{g}|}$, which is now a function of the vierbein; no conditions are imposed on the spin-connection.

☐ Varying around the metric compatible and torsionless (Levi-Civita) spin-connection, one finds the analogue of (11.8)

11.11
$$\Sigma_{\mu}{}^{a}{}_{b} = \Sigma_{\mu}{}^{a}{}_{b}|_{\mathrm{LC}} + A_{\mu}\delta^{a}_{b}$$

For the spin-connection to become the *Levi-Civita spin-connection*, one must set the gauge parameter A_μ to zero. Equivalently, one may impose $\Sigma_{\mu}{}^{a}{}_{a} = 0$, which is the *metricity condition*, since metric compatibility is equivalent to the anti-symmetry of the spin connection in its Latin indices; see Tetrad Postulate.

☐ The vierbein Palatini action (11.10), or rather its extension to the *Einstein-Cartan theory*, is essential in the formulation of a generally covariant fermionic action which couples fermions to gravity. Since fermions are sources of torsion, this requires a theory in which *torsion* is allowed, but *metricity* preserved. In this respect, the Einstein-Cartan theory is a natural extension of GRT when fermions are involved.

Chapter 12

Einstein-Cartan Theory

12.1 Riemann-Cartan Geometry [AT,FH]

☐ The *Einstein–Cartan theory* of gravity (ECT) is a modification of GRT, allowing spacetime to have *torsion* in addition to *curvature*. This theory, also known as the *Einstein–Cartan–Kibble–Sciama* (ECKS) theory, has a long history with major contributions by Einstein, Cartan, Utimaya, Kibble and Sciama. Élie Cartan initiated this development in 1922 by reformulating general relativity on the basis of a new mathematical construct, now known as *Riemann–Cartan (RC) geometry*. The main difference with Riemannian geometry is that in the latter the affine connection is derived from the metric, whereas the primary structure characterizing the RC spacetime manifold, commonly denoted $\mathcal{U}_4$, is the independent pair $\{\mathbf{e}^a, \boldsymbol{\omega}^a{}_b\}$ of *frame field* and *Maurer-Cartan connection* .

☐ In modern view, ECT is the simplest version of a *Poincaré Gauge Theory* (PGT). This class of theories within the Riemann-Cartan geometry describes gravitational interactions as a gauge theory based on the *Poincaré group* $\mathrm{P}(1,3)$, i.e., the symmetry group underlying special relativity consisting of translations $\mathrm{T}(4)$ and Lorentz rotations $\mathrm{SO}(1,3)$. In PGT, these translational and rotational symmetries are 'gauged' to become local, inducing two sets of *gauge potentials* $\left\{e_\mu{}^a, \Sigma_\mu{}^{ab}\right\}$: the vierbein field and its inverse associated with translations; the spin (Lorentz) connection associated with Lorentz transformations. These fields are used to define a coderivative D_μ thereby ensuring that the theory remains invariant under local Poincaré transformations; see Gauge Principle and [Wikipedia: Poincaré gauge theory].

☐ The corresponding 'gauge field strengths' tensors are obtained by commutation with the coderivative and found to be *torsion* and *curvature* as defined in (4.24) and (6.3) equivalent to, respectively, the First Cartan Equation (4.25) and Second Cartan Equation (10.6):

12.1

$$T^a_{\mu\nu} = 2D^{\Sigma}_{[\mu}e^a_{\nu]} \qquad \mathbf{T}^a = \mathrm{d}\mathbf{e}^a + \boldsymbol{\omega}^a{}_b \wedge \mathbf{e}^b$$

12.2

$$R^{ab}_{\mu\nu} = 2D^{\Sigma}_{[\mu}\Sigma^{ab}_{\nu]} \qquad \boldsymbol{\mathcal{R}}^{a}{}_{b} = \mathrm{d}\boldsymbol{\omega}^{a}{}_{b} + \boldsymbol{\omega}^{a}{}_{c} \wedge \boldsymbol{\omega}^{c}{}_{b}$$

Thus, gauging the Poincaré group leads directly to the RC geometry of spacetime. Notable is the correspondence between *torsion* and *translation symmetry*.

☐ In the EC theory the curvature 2-form $\boldsymbol{\mathcal{R}}^{a}{}_{b}$ differs from the Riemann 2-form $\boldsymbol{R}^{a}{}_{b}$ through the torsion dependence of Maurer-Cartan connection. The curvature tensor still satisfies $\mathcal{R}_{abcd} = \mathcal{R}_{[ab][cd]}$. However, in the RC spacetime the Ricci tensor $\mathcal{R}^{a}{}_{b} := \mathcal{R}^{ac}_{bc}$, although well-defined, is *not necessarily symmetric* because the Cartan curvature has not the symmetry (10.4c) of the Riemann. When torsion vanishes, the Cartan and Riemann curvature tensors coincide.

12.2 Einstein-Cartan Action [CC,JTW]

☐ In the Einstein-Cartan theory the *Cartan curvature tensor* (12.2) plays central role in the construction of the action. The first step to derive this action is to bring the Einstein-Hilbert action (11.6), with the Ricci scalar replaced by the Riemann–Cartan curvature scalar $\mathcal{R}$, into the form of a *directed integral*. This may be accomplished by inserting the product of the unit volume element and its inverse (A.1):

12.3

$$S_{\mathrm{EC}} := \frac{1}{2\kappa}\int d^4x\sqrt{|g|}\mathrm{I}_4\,\mathrm{I}^4\,\mathcal{R} = \frac{1}{2\kappa}\int dx_4\,\mathrm{I}^4\,\mathcal{R}$$

The volume element $dx_4 := d^4x\sqrt{|g|}\mathrm{I}_4$ is the *signed* invariant volume measure implying a choice of orientation on $\mathcal{M}$; see Integration Measure.

☐ Subsequently, the definition of the Riemann-Cartan curvature scalar in the integrand may be reworked into

12.4

$$\mathcal{R} := \mathcal{R}^{cd}_{cd} = \frac{1}{2}\delta^{mn}_{cd}\mathcal{R}^{cd}_{mn}$$

The pre-factor is the generalized Kronecker-delta symbol (B.1) for $p = 2$. The first identity (B.4), together with the tetrad form (A.2) of the inverse unit volume and formula (10.5) for the tetrad curvature 2-form, then allows the Einstein-Cartan action, with cosmological constant, to be cast in the *tetradic Einstein-Cartan form*:

12.5

$$S_{\mathrm{EC}}[\mathbf{e},\boldsymbol{\omega}] = \frac{1}{4\kappa}\int dx_4\,\varepsilon_{abcd}\mathbf{e}^a \wedge \mathbf{e}^b \wedge \left(\boldsymbol{\mathcal{R}}^{cd}[\boldsymbol{\omega}] + \frac{\Lambda}{6}\mathbf{e}^c \wedge \mathbf{e}^d\right)$$

The Cartan curvature $\boldsymbol{\mathcal{R}}^{a}{}_{b}$ is the functional of $\boldsymbol{\omega}^{a}{}_{b}$ as given in (12.2).

□ Without embellishments, Einstein-Cartan theory is a viable modification of Einstein-Hilbert gravitation, the frame field and the connection having the dynamical interpretation of T(4) and SO(1,3) gauge potentials. Moreover, the theory is completely consistent with all observational tests of gravity so far. For these reasons the Einstein-Cartan formulation is sometimes preferred over the canonical Einstein-Hilbert formulation e.g. in explorations of quantum gravity. Significant departures from the Einstein theory are expected only for matter densities well beyond the nuclear threshold.

12.3 Equation of Motion [CC,JTW]

□ The field equations in the Einstein-Cartan theory are obtained from the tetradic action (12.5) by performing *independent stationary variations* with respect to the gravitational gauge fields, i.e. the frame field $\{\mathbf{e}^a\}$ and the Lorentz connection $\{\boldsymbol{\omega}^{ab}\}$. The variation with respect to the frame field yields:

12.6

$$\delta_{\mathbf{e}} S_{\mathrm{EC}}[\mathbf{e}, \boldsymbol{\omega}] = \frac{1}{2\kappa} \int dx_4\, \varepsilon_{abcd} \delta \mathbf{e}^a \wedge \mathbf{e}^b \wedge \left(\boldsymbol{\mathcal{R}}^{cd} + \frac{\Lambda}{3} \mathbf{e}^c \wedge \mathbf{e}^d \right)$$

The condition that the action vanishes for arbitrary variations $\delta \mathbf{e}^a$ gives the equation of motion

12.7

$$\mathbf{G}_a := \frac{1}{2} \varepsilon_{abcd} \mathbf{e}^b \wedge \left(\boldsymbol{\mathcal{R}}^{cd} + \frac{\Lambda}{3} \mathbf{e}^c \wedge \mathbf{e}^d \right) = 0$$

which is the *Einstein (vacuum) field equation* in the RC tetrad formalism, including a cosmological constant.

12.8 Einstein Tensor

a. Expand the curvature 2-form $\boldsymbol{\mathcal{R}}^{cd}$ in (12.7) to create the *Einstein 3-form*, also called the moment of rotation 3-form:

$$\mathbf{G}_a = \frac{1}{4} \varepsilon_{abcd} \mathbf{e}^b \wedge \mathbf{e}^m \wedge \mathbf{e}^n \left(\mathcal{R}^{cd}_{mn} + \frac{\Lambda}{3} \delta^{cd}_{mn} \right)$$

b. Multiply from the left with the pseudoscalar I_4. Then use (C.2) and (B.4) for $p = 3$ to get the dual the 1-form:

$$*\mathbf{G}_a = -\frac{1}{4} \delta^{bmn}_{acd} \left(\mathcal{R}^{cd}_{mn} + \frac{\Lambda}{3} \delta^{cd}_{mn} \right) \mathbf{e}_b$$

c. Expand the generalized Kronecker delta's with (B.1) to obtain the tensor form of the EC vacuum field equations:

$$*\mathbf{G}_a = \left(\mathcal{G}^b{}_a - \Lambda\delta^b_a\right)\mathbf{e}_b = 0$$

The Einstein tensor $\mathcal{G}_{ba} = \mathcal{G}_{(ab)} - \mathcal{R}_{[ab]}$ is *asymmetric* due to the asymmetry of the Ricci tensor when torsion is present.

12.4 Torsion Equation [AT,CC]

☐ In the Einstein-Cartan theory of gravity, one usually assumes *metric compatibility* of the Lorentz (spin) connection, i.e. $\boldsymbol{\omega}_{ab} = -\boldsymbol{\omega}_{ba}$. In that case, also assuming that the variation δ and the exterior derivative d commute, one concludes from eq (10.6) that the variation of the curvature 2-form is the *exterior coderivative* of the variation of the connection: $\delta\boldsymbol{\mathcal{R}}^{ab} = \mathrm{D}\delta\boldsymbol{\omega}^{ab}$. Hence

12.9

$$\delta_{\boldsymbol{\omega}} S_{\mathrm{EC}}[\mathbf{e},\boldsymbol{\omega}] = \frac{1}{4\kappa}\int dx_4\ \varepsilon_{abcd}\mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathrm{D}\delta\boldsymbol{\omega}^{cd}$$

It should be remarked that the variation $\delta\boldsymbol{\omega}^{ab}$ transforms as a tensor, although the connection itself does not; see (4.10). Thus, the coderivative in (12.9) is well-defined.

☐ An integration by parts may be performed with the help of the Leibniz rule

12.10

$$\begin{aligned}&\varepsilon_{abcd}\mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathrm{D}\delta\boldsymbol{\omega}^{cd} = \\ &\mathrm{D}(\varepsilon_{abcd}\mathbf{e}^a \wedge \mathbf{e}^b \wedge \delta\boldsymbol{\omega}^{cd}) - 2\varepsilon_{abcd}\mathbf{T}^a \wedge \mathbf{e}^b \wedge \delta\boldsymbol{\omega}^{cd}\end{aligned}$$

where $\mathbf{T}^a$ is the torsion 1-form as defined in (4.26). Since in (12.10) the exterior coderivative at the right-hand side acts on a scalar, this D may be replaced by just the exterior derivative d. Ignoring the resulting boundary term, one obtains the *torsion field equation* of the (vacuum) EC theory:

12.11

$$\mathbf{S}_{ab} := -\varepsilon_{abcd}\mathbf{T}^c \wedge \mathbf{e}^d = 0$$

☐ When the tetrad is assumed to be invertible, one is led to the equation $\mathbf{T}^a = \mathrm{D}\mathbf{e}^a = 0$. Hence, the vanishing torsion condition (4.27) of GRT is achieved *dynamically* in Einstein-Cartan gravity. Even when a vanishing torsion is not imposed as a constraint, the theory defined by the Einstein-Cartan action is equivalent to standard GRT, at least in the vacuum case.

12.12 Modified Torsion Tensor

a. The torsion equation in tensor components may be obtained by expanding the torsion 2-form $\mathbf{T}^c = (1/2)T^c_{ab}\mathbf{e}^a \wedge \mathbf{e}^b$ in (12.11) and then by constructing the dual; see Equation of Motion.

b. Next one needs (B.1) and (B.4) for $p = 3$ to find:

$$*\mathbf{S}_{ab} = \left(T^c_{ab} + 2\delta^c_{[a} T^d_{b]d} \right) \mathbf{e}_c := S^c_{ab} \mathbf{e}_c = 0$$

The tensor S^c_{ab} defined here is called the *modified* torsion tensor; it differs from the torsion tensor in its trace.

12.5 Bianchi Identities [AT,FH]

☐ By construction the action (12.5) of the Einstein-Cartan theory is invariant under *local Lorentz transformations* (LLTs). The action is also invariant under *manifold diffeomorphisms* (MDs), i.e. smooth, invertible mappings $\mathcal{M}_4 \to \mathcal{M}_4$. The latter symmetry derives from the general covariance under a change of coordinates. These two transformations comprise the set of *fundamental symmetries* of general relativity. They may be seen as *gauge symmetries* of the theory, where it should be noted that diffeomorphism invariance of ECT is closely related to the symmetry group of local spacetime translations $T(4)$; see Riemann-Cartan Geometry.

☐ The two fundamental symmetries are continuous symmetries. So, by Noether's theorem, one expects two conservation laws. In both GRT and ECT they take the form of two sets of (contracted) *Bianchi identities* intrinsic to the geometrical structure of the GR and RC spacetimes. In ECT these identities are $\mathrm{D}\mathbf{T}^a = \boldsymbol{\mathcal{R}}^a{}_b \wedge \mathbf{e}^b$, $\mathrm{D}\boldsymbol{\mathcal{R}}^a{}_b = 0$; see (10.9),(10.13) for their Riemannian equivalents. They may be used to evaluate the exterior coderivatives of the Einstein and torsion 3-forms $\mathbf{G}_a$ (12.7) and $\mathbf{S}_{ab}$ (12.11):

12.13

$$\mathrm{D}\mathbf{G}_a = \frac{1}{2}\varepsilon_{abcd}\mathbf{T}^b \wedge \boldsymbol{\mathcal{R}}^{cd} - \Lambda\mathbf{T}^b \wedge \boldsymbol{\eta}_{ab}$$

12.14

$$\mathrm{D}\mathbf{S}_{ab} = \boldsymbol{\eta}_{ac} \wedge \boldsymbol{\mathcal{R}}_b{}^c - \boldsymbol{\eta}_{bc} \wedge \boldsymbol{\mathcal{R}}_a{}^c$$

In the derivation of the last equation identity (E.2d) has been used.

☐ Equation (12.13) is the generalization to RC space, of the twice contracted Bianchi identity (10.22) which is crucial in standard GRT. It is interpreted to be a consequence of *diffeomorphism invariance*. The second equation derives from Lorentz symmetry and relates to the *conservation of angular momentum*.

12.15 GRT

a. Without torsion (12.13) reduces to the GRT Einstein equation $\mathrm{D}\mathbf{G}_a = 0$.
b. Proof: the dual of equation (12.8c) gives the *Einstein 3-form*

$$\mathbf{G}_a = \left(G^b{}_a - \Lambda\delta^b_a \right) \boldsymbol{\eta}_b$$

expanded with respect to the basis 3-form $\boldsymbol{\eta}_b$ defined in (C.3.b).

c. Since $D\boldsymbol{\eta}_b = 0$ (no torsion), one obtains with (E.2a):

$$- * \,\mathrm{D}\mathbf{G}_a = D_b\left(G^b{}_a - \Lambda\delta^b_a\right) = 0$$

which is the tetrad form of the covariant conservation equation (10.24) of the (symmetric) Einstein tensor (10.23).

12.6 Noether Identities [AT,FH,CC]

□ Noether's (first) theorem states that symmetries of the action are associated with *conserved quantities*. In the usual procedure the conservation laws are obtained by implementing the symmetries (in the present case diffeomorphisms and local Lorentz transformations) in the action. The result would then imply the Bianchi identities (12.13,14). However, since these Bianchi identities have already been established, the shorter alternative is to derive the conservation laws therefrom. By some algebraic manipulations one then obtains the differential *Noether identities* of the Einstein-Cartan theory:

12.16

$$\mathrm{D}\mathbf{G}_a = \mathbf{T}^b_a \wedge \mathbf{G}_b - \frac{1}{2}\boldsymbol{\mathcal{R}}^{bc}_a \wedge \mathbf{S}_{bc}$$

12.17

$$\mathrm{D}\mathbf{S}_{ab} = \mathbf{e}_b \wedge \mathbf{G}_a - \mathbf{e}_a \wedge \mathbf{G}_b$$

□ These equations look much like *balance equations* for energy-momentum and angular momentum; this will be further explored in Conservation Laws. For vanishing torsion, GRT is recovered as shown in (12.15). In this limit equation (12.17) enforces the symmetry of the Einstein tensor which is conserved on account of (12.16). As an aside it may be is mentioned that in the limit of zero torsion and curvature the above Noether equations reduce to the conservation laws of energy-momentum and angular momentum in Minkowski spacetime.

12.18 Notes

a. The 1-form fields appearing in (12.16) are defined as the contractions

$$\mathbf{T}^a_b := \mathbf{e}_b \cdot \mathbf{T}^a = T^a_{bc}\mathbf{e}^c \quad \boldsymbol{\mathcal{R}}^{ab}_c := \mathbf{e}_c \cdot \boldsymbol{\mathcal{R}}^{ab} = \mathcal{R}^{ab}_{cd}\mathbf{e}^d$$

b. Since $\mathbf{G}_a$ is a 3-form, $(\mathbf{T}^b \wedge \mathbf{G}_b)$ is a 5-form, hence vanishes identically. By contracting with $\mathbf{e}_a$ and working out the inner product with the algebraic rule (E.1), one finds that the right-hand sides of (12.13) and (12.16) are equivalent.

c. To arrive at (12.17), one may use (E.2d) and the auxiliary formula

$$\mathbf{e}^k \wedge \mathbf{G}_a = -\frac{1}{2}\left(\boldsymbol{\eta}_{ab} \wedge \boldsymbol{\mathcal{R}}^{bk} + \boldsymbol{\eta}_{ca} \wedge \boldsymbol{\mathcal{R}}^{kc} + \delta^k_a\boldsymbol{\eta}_{bc} \wedge \boldsymbol{\mathcal{R}}^{bc}\right)$$

12.7 BR Symmetrization [FH,JTW]

☐ The Noether identities (12.16,17) may be brought in *component form* by substituting the representations, see (12.8,12),

12.19

$$\mathbf{G}_a = \left(\mathcal{G}^b{}_a - \Lambda\delta^b_a\right)\boldsymbol{\eta}_b \quad \mathbf{S}_{ab} = S^c_{ab}\boldsymbol{\eta}_c$$

With the help of of formulae (A.2),(B.4) and C3.b), one finds that the exterior derivative $\mathrm{D} = \mathbf{e}^c \wedge D_c$ of $\boldsymbol{\eta}_a$ is proportional to the trace of the torsion tensor:

12.20

$$\mathrm{D}\boldsymbol{\eta}_a = \mathbf{T}^b \wedge \boldsymbol{\eta}_{ab} = -T^b_{ab}\mathrm{I}^4$$

This trace vanishes if the torsion tensor is *completely anti-symmetric* which is e.g. the case for the Dirac field; see ECD Gravity.

☐ These manipulations result in ten differential *Noether (Bianchi) identities* involving the Einstein tensor $\mathcal{G}^b{}_a$ and modified torsion tensor S^c_{ab}:

12.21

$$-*\mathrm{D}\mathbf{G}_a = \overset{*}{D}_b\,\mathcal{G}^b{}_a = T^c_{ab}\mathcal{G}^b{}_c - \frac{1}{2}\mathcal{R}^{cd}_{ab}S^b_{cd}$$

12.22

$$-*\mathrm{D}\mathbf{S}_{ab} = \overset{*}{D}_c\,S^c_{ab} = \mathcal{G}_{ba} - \mathcal{G}_{ab} = -2\mathcal{R}_{[ab]}$$

with derivative operator $\overset{*}{D}_a := D_a + T^b_{ab}$.

☐ It is seen that the divergence of modified torsion tensor is directly related to the *anti-symmetric* part of the Ricci tensor or, what amounts to the same, the Einstein tensor $\mathcal{G}_{ab}$. Hence, *local Lorentz symmetry* of the EC-action leads to Ricci and Einstein tensors *symmetric* in their indices:

12.23

$$G_{ab} := \mathcal{G}_{(ab)} = \mathcal{G}_{ab} + \frac{1}{2}\overset{*}{D}_c\,S^c_{ab}$$

This is entirely similar to the well-known Belinfante-Rosenfeld (BR) construction of a symmetric energy-momentum tensor in classical or quantum field theory. [Wikipedia: Belinfante–Rosenfeld stress–energy tensor]

Chapter 13

ECKS Theory

13.1 Matter Action [FH,CC]

☐ By the addition of a *matter action* $S_{\mathrm{EC}}[\mathbf{e},\boldsymbol{\omega}] \to S_{\mathrm{EC}}[\mathbf{e},\boldsymbol{\omega}] - S_{\mathrm{M}}[\mathbf{e},\boldsymbol{\omega},\psi]$ to the EC-action (12.5), both EC field equations (12.7) and (12.11) acquire a source term depending on $\{\mathbf{e}^a, \boldsymbol{\omega}^{ab}\}$ as well as on one or more matter fields ψ:

13.1

$$\mathbf{G}_a = \kappa \frac{\delta S_{\mathrm{M}}[\mathbf{e},\boldsymbol{\omega},\psi]}{\delta \mathbf{e}^a} := \kappa \boldsymbol{\tau}_a$$

13.2

$$\mathbf{S}_{ab} = 2\kappa \frac{\delta S_{\mathrm{M}}[\mathbf{e},\boldsymbol{\omega},\psi]}{\delta \boldsymbol{\omega}^{ab}} := \kappa \boldsymbol{\sigma}_{ab}$$

These equations, first proposed by Tom Kibble (1961) and Dennis Sciama (1962), are the field equations of the *ECKS-theory of gravity.* This is the special case of a gauge theory which has the curvature of the Riemann-Cartan spacetime as gravitational action. Sciama and Kibble localized the Poincaré group $\mathrm{P}(1,3)$ of spacetime symmetries and in this way established that gravity can consistently described as a gauge theory.

☐ In Equation of Motion it has been demonstrated that the Einstein 3-form $\mathbf{G}_a$ is equivalent to an *asymmetric* Einstein tensor including cosmological constant. Using representation (12.15b), one may then cast equation (13.1) in the form of the (generalized) *Einstein equation*

13.3

$$\mathbf{G}_a = \left(\mathcal{G}^b{}_a - \Lambda \delta^b_a\right) \boldsymbol{\eta}_b = \kappa \tau^b{}_a \boldsymbol{\eta}_b$$

This equation shows that the 3-form $\boldsymbol{\tau}_a$ is the source of curvature. It may be identified as the *canonical* energy-momentum density of matter. The *energy-momentum tensor* defined through $\boldsymbol{\tau}_a := \tau^b{}_a \boldsymbol{\eta}_b$, is *asymmetric* like the Ricci and Einstein tensors.

☐ By a similar reasoning, and the use of (12.12), the torsion equation (13.2) can be rewritten into

13.4

$$\mathbf{S}_{ab} = \left(T^c_{ab} + 2\delta^c_{[a}T^d_{b]d}\right)\boldsymbol{\eta}_c = \kappa\sigma^c_{ab}\boldsymbol{\eta}_c$$

Analogously to curvature being sourced by the energy-momentum density of the matter sources, torsion is sourced by the spin angular momentum density $\boldsymbol{\sigma}_{ab} =: \sigma^c_{ab}\boldsymbol{\eta}_c$ of matter. The torsion equation can be solved to give the so-called *Cartan equations*

13.5

$$T^c_{ab} = \kappa\left(\sigma^c_{ab} + \frac{1}{2}\delta^c_a\sigma^d_{bd} + \frac{1}{2}\delta^c_b\sigma^d_{da}\right)$$

These equations are linear and algebraic which implies that, outside regions with spin densities, torsion vanishes, or stated differently, *torsion does not propagate.* If the matter is spinless altogether, torsion vanishes identically and the equation of motion (13.3) reduces to the Einstein field equation of GRT.

13.2 Einstein Equation [FH,JTW]

☐ Because of the linear structure of equations (13.1,2), the pair of Noether identities (12.16,17) immediately implies the *conservation laws* of *energy-momentum* and *angular momentum* in the ECKS theory:

13.6

$$\mathrm{D}\boldsymbol{\tau}_a = \mathbf{T}^b_a \wedge \boldsymbol{\tau}_b - \frac{1}{2}\mathbf{R}^{bc}_a \wedge \boldsymbol{\sigma}_{bc}$$

13.7

$$\mathrm{D}\,\boldsymbol{\sigma}_{ab} = 2\mathbf{e}_{[b} \wedge \boldsymbol{\tau}_{a]}$$

☐ Under the assumption that the torsion tensor is completely anti-symmetric, these equations read in tensor form, see (12.21,22):

13.8

$$D_b\tau^b{}_a = T^c_{ab}\tau^b{}_c - \frac{1}{2}R^{cd}_{ab}\sigma^b_{cd}$$

13.9

$$D_c\sigma^c_{ab} = \tau_{ba} - \tau_{ab}$$

They are ten covariant *Riemann-Cartan conservation laws* for the canonical energy-momentum $\tau^b{}_a$ and spin angular momentum σ^c_{ab} tensors.

☐ The divergence of the spin current (13.9) is directly related to the anti-symmetric part of the canonical energy momentum tensor. Analogously to (12.23), this allows a Belinfante-Rosenfeld (BR) construction of a symmetric energy-momentum tensor of matter

13.10

$$t_{ab} := \tau_{(ab)} = \tau_{ab} + \frac{1}{2} D_c \sigma^c_{ab}$$

Thus, local conservation of angular momentum (13.9) leads to the *Einstein equation* of the ECKS theory

13.11

$$G_{ab} - \Lambda\delta_{ab} = \kappa\, t_{ab}$$

with both Einstein tensor and matter energy-momentum tensor *symmetric* in their indices. [Wikipedia: Belinfante–Rosenfeld stress–energy tensor]

13.3 ECD Gravity [FH,JTW]

☐ In the ECKS-theory, torsion is sourced by the spin density of matter; see (13.2). In this context, a much studied model is the free Dirac fermionic field of spin and mass m described by a Dirac spinor ψ minimally coupled to torsion. The matter Lagrangian density of this *Einstein-Cartan-Dirac (ECD) theory* is given by:

13.12

$$\mathcal{L}_{\mathrm{D}} := \frac{1}{2}\left[\bar{\psi}(i\gamma^a \mathcal{D}_a\psi - m\psi) - (i\bar{\mathcal{D}}_a\bar{\psi}\gamma^a + m\bar{\psi})\psi\right]$$

where $\bar{\psi} := \psi^\dagger\gamma^0$ is the adjoint spinor, the dagger indicating the Hermitian conjugate. The Dirac matrices $\{\gamma^a; a = 0, 1, 2, 3\}$ have their special relativistic values and are considered to be constant; see Dirac Algebra.

☐ The Lagrangian is Hermitian and of first order, i.e. only a first derivative of the fields enters, which in this case is the Fock-Ivanenko coderivative:

13.13

$$\mathcal{D}_a\psi := \partial_a\psi + \Gamma_a\psi \quad \bar{\mathcal{D}}_a\bar{\psi} := (\mathcal{D}_a\psi)^\dagger\gamma^0 = \partial_a\bar{\psi} - \bar{\psi}\,\Gamma_a$$

The Spinor Connection is built from the *spin connection*:

13.14

$$\mathcal{D}_a := e^\mu{}_a\left(\partial_\mu + \frac{1}{2}\Sigma_{\mu bc}\sigma^{bc}\right)$$

and the *generators* of the spinor algebra:

13.15

$$\sigma^{bc} := \frac{1}{2}\gamma^b \wedge \gamma^c = \frac{1}{4}[\gamma^b, \gamma^c] \quad (\sigma^{bc})^\dagger = -\gamma^0\sigma^{bc}\gamma^0$$

☐ The corresponding Hermitian matter action in ECD gravity is

13.16

$$S_{\mathrm{M}}[e,\Sigma,\psi]=\int d^4x\sqrt{|g|}\mathcal{L}_{\mathrm{D}}(x)$$

By construction it is a functional of the Dirac spinors and the *vierbein gravitational potentials* $\left\{e_\mu{}^a,\Sigma_\mu{}^{ab}\right\}$. Varying the action with respect to $\bar{\psi}$, or ψ, and setting the result equal to zero, one obtains the Dirac equation (6.19) or its adjoint.

13.4 Tetradic Dirac Action [CC,FH]

☐ By the same procedure by which the tetradic Einstein-Cartan form (12.5) is obtained from (11.6), one may rewrite the action (13.16) into a functional of the Cartan potentials $\{\mathbf{e}^a,\boldsymbol{\omega}^{ab}\}$. First, the Lagrangian (13.12) is redefined like in (12.3):

13.17

$$S_{\mathrm{D}}[\mathbf{e},\omega,\psi]=\int dx_4\mathrm{I}^4\mathcal{L}_{\mathrm{D}}(x):=\int dx_4L_{\mathrm{D}}(\psi,\mathcal{D}\psi,x)$$

☐ Next, one uses the identities (A.2) and (B.3) for $p=1$ to derive:

13.18

$$\mathrm{I}^4\gamma^a\mathcal{D}_a=\mathrm{I}^4\gamma^a\delta_a^b\mathcal{D}_b=\boldsymbol{\gamma}\wedge\mathcal{D}\quad\boldsymbol{\gamma}:=\gamma^a\boldsymbol{\eta}_a$$

where $\boldsymbol{\eta}_a$ is the 3-form (C.3b); the (coordinate free) *gradient coderivative* is defined as:

13.19

$$\mathcal{D}:=\mathbf{e}^a\mathcal{D}_a=\nabla+\frac{1}{2}\boldsymbol{\omega}_{bc}\sigma^{bc}=\nabla+\frac{1}{2}\boldsymbol{\omega}^{bc}\sigma_{bc}$$

☐ The result is the *tetradic form* of the Dirac Lagrangian in Riemann-Cartan space:

13.20

$$L_{\mathrm{D}}(\psi,\mathcal{D}\psi,x)=\frac{1}{2}i\left(\bar{\psi}\boldsymbol{\gamma}\wedge\mathcal{D}\psi-\bar{\mathcal{D}}\bar{\psi}\wedge\boldsymbol{\gamma}\psi\right)-\eta m\bar{\psi}\psi$$

which is minimally coupled to the Riemann-Cartan spacetime via the gauge potentials $\mathbf{e}^a$ (contained in $\boldsymbol{\gamma}$ and $\eta=\mathrm{I}^4$) and $\boldsymbol{\omega}^{ab}$ (contained in $\mathcal{D}$). The potentials enter the Lagrangian *linearly.*

☐ Because of this linearity, it is now an easy exercise to calculate from (13.20) the source term of (13.2). This yields the torsion equation of the ECD-theory (13.4), with the tensor components of the *spin density* at the right-hand side given by

13.21

$$\sigma_{ab}^c=\frac{1}{2}i\bar{\psi}(\gamma^c\wedge\sigma_{ab}+\sigma_{ab}\wedge\gamma^c)\psi=\frac{1}{2}i\bar{\psi}\left\{\gamma^c,\sigma_{ab}\right\}\psi$$

The *Dirac spin tensor* is *totally* antisymmetric in its indices; see (13.27).

13.5 Energy-Momentum Tensor [CC,FH]

☐ To obtain the source term $\boldsymbol{\tau}_a$ at the right-hand side of (13.1), the Dirac Lagrangian density (13.20) is varied with respect to the potentials $\{\mathbf{e}^a\}$. The definitions (C.3) imply: $\delta\eta = -\delta\mathbf{e}^a \wedge \boldsymbol{\eta}_a$ and $\delta\boldsymbol{\eta}_b = -\delta\mathbf{e}^a \wedge \boldsymbol{\eta}_{ab}$, which then directly leads to the source term:

13.22

$$\boldsymbol{\tau}_a = \frac{1}{2}i\left(\bar{\mathcal{D}}\bar{\psi} \wedge \gamma^b\boldsymbol{\eta}_{ab}\psi - \bar{\psi}\gamma^b\boldsymbol{\eta}_{ab} \wedge \mathcal{D}\psi\right) + \boldsymbol{\eta}_a m\bar{\psi}\psi$$

☐ With the identity (E.3b) and the definition of the coderivative (13.19) this may be reworked into the sum of two contributions

13.23

$$\begin{aligned}\boldsymbol{\tau}_a = &\frac{1}{2}\left[\left(i\bar{\mathcal{D}}_b\bar{\psi}\gamma^b\psi + m\bar{\psi}\psi\right) - \left(i\bar{\psi}\gamma^b\mathcal{D}_b\psi - m\bar{\psi}\psi\right)\right]\boldsymbol{\eta}_a \\ &+\frac{1}{2}i\left(\bar{\psi}\gamma^b\mathcal{D}_a\psi - \bar{\mathcal{D}}_a\bar{\psi}\gamma^b\psi\right)\boldsymbol{\eta}_b\end{aligned}$$

The terms in the first line vanish if the spinor fields are required to satisfy the Dirac equations

13.24

$$i\gamma^b\mathcal{D}_b\psi - m\psi = 0 \quad i\bar{\mathcal{D}}_b\bar{\psi}\gamma^b + m\bar{\psi} = 0$$

☐ This leaves the last term of (13.23) with components of the on-shell (canonical) Dirac energy-momentum tensor $\boldsymbol{\tau}_a := \tau^b{}_a\boldsymbol{\eta}_b$ given by:

13.25

$$\tau^b{}_a = \frac{1}{2}i\left(\bar{\psi}\gamma^b\mathcal{D}_a\psi - \bar{\mathcal{D}}_a\bar{\psi}\gamma^b\psi\right)$$

This expression is obviously not symmetric in its indices. The existence of a spin current demands the canonical energy-momentum tensor to have an anti-symmetric part so as to fulfill the requirement of local angular momentum conservation; see: Conservation Laws.

13.6 Belinfante-Rosenfeld Tensor [FH,JTW]

☐ In algebraic manipulations, gamma-matrices may be treated like vectors in GA, e.g.

13.26

$$\begin{aligned}\gamma^c(\gamma^a \wedge \gamma^b) &= \gamma^c \cdot (\gamma^a \wedge \gamma^b) + \gamma^c \wedge \gamma^a \wedge \gamma^b \\ &= \eta^{ca}\gamma^b - \eta^{cb}\gamma^a + \gamma^a \wedge \gamma^b \wedge \gamma^c\end{aligned}$$

With definition (D.2) of the spin matrices, it is then easily shown that the spin-density tensor of the Dirac field (13.21) can be written in the form:

13.27
$$\sigma^{abc} = \frac{1}{2} i\bar{\psi} \left\{ \sigma^{ab}, \gamma^c \right\} \psi = \frac{1}{2} i\bar{\psi} \left(\gamma^a \wedge \gamma^b \wedge \gamma^c \right) \psi$$

which makes it evident that the spin-density tensor is *totally antisymmetric* in its indices.

□ A relation with the *antisymmetric* part of the energy-momentum tensor (13.25) is established through the divergence

13.28
$$\mathcal{D}_c \sigma^{cab} := \frac{1}{2} i \left[\left(\bar{\mathcal{D}}_c \bar{\psi} \right) \gamma^a \wedge \gamma^b \wedge \gamma^c \psi + \bar{\psi} \gamma^a \wedge \gamma^b \wedge \gamma^c \left(\mathcal{D}_c \psi \right) \right]$$

A straightforward calculation with the help of the identity (13.26) and use of the Dirac equations (13.24) gives the on-shell result:

13.29
$$\mathcal{D}_c \sigma^{cab} = \mathcal{D}_c \sigma^{abc} = \tau^{ba} - \tau^{ab} = -2\tau^{[ab]}$$

in complete agreement with the general conservation law (13.9).

□ In the present case of the Dirac field, the BR construction (13.10) amounts to simply symmetrizing the canonical energy-momentum tensor $t^{ab} = \tau^{(ab)}$. It works out this way because in this particular case the spin-density tensor is totally antisymmetric. [Wikipedia: Belinfante–Rosenfeld stress–energy tensor]

13.7 Contorsion Tensor [CC,JTW]

□ The *contorsion tensor* in differential geometry is the difference between a connection with and without torsion, e.g. the difference between the general asymmetric affine connection $\Gamma^{\lambda}_{\mu\nu}$ and the symmetric Levi-Civita connection (1.29). Specifically, the contorsion tensor C_{abc} in Riemann-Cartan space is defined as the difference between the spin connection (Ricci coefficient) (4.20) and the torsion free connection (4.30), here indicated by the overhead symbol ∘:

13.30
$$\omega_{abc} = \overset{\circ}{\omega}_{abc} + C_{abc}$$

Metric compatibility is assumed, i.e. both connections and the contorsion tensor are *anti-symmetric* in their last two indices.

□ The explicit form of C_{abc} may be obtained from the tensor form (4.24) of the first Cartan equation which reads in tetrad base components:

13.31
$$2\omega_{[ab]c} = f_{abc} - T_{cab}$$

It differs from (4.30) by the components of the torsion 2-form:

13.32

$$\mathbf{T}^a = \frac{1}{2} T^a{}_{bc} \mathbf{e}^b \wedge \mathbf{e}^c \quad T^a{}_{bc} = 2\Gamma^a{}_{[bc]}$$

Combining three index permutations of (13.31) in the same way as described under (4.30), one derives for the contorsion tensor the invertible relation:

13.33

$$C_{abc} = \frac{1}{2}\left(T_{abc} + T_{cba} - T_{bca}\right) = \frac{1}{2} T_{abc} + T_{[cb]a}$$

□ A contraction with $\mathbf{e}^a$ on the first index of (13.30) yields the Maurer-Cartan connection as the sum of the Levi-Civita and contorsion 1-forms:

13.34

$$\boldsymbol{\omega}^a{}_b = \overset{\circ}{\boldsymbol{\omega}}{}^a{}_b + \mathbf{C}^a{}_b \quad \mathbf{C}^a{}_b := \mathbf{e}^c C_c{}^a{}_b$$

This decomposition of the connection is unique, because the torsion-free spin connection satisfies the torsion constraint (4.27). This is easily shown by wedging with $\mathbf{e}^b$ and using the identity $\mathbf{C}^a{}_b \wedge \mathbf{e}^b = \mathbf{T}^a$.

13.8 Hehl-Datta Equation [FH]

□ Torsion enters ECD gravity through the coderivative (13.13) of the Dirac spinor in the Dirac Lagrangian (13.12). By separating off the torsion-free spin connection as in (13.30), one may decompose the Lagrangian into a part with the torsion-free (Levi-Civita) coderivative and a remainder which is a contraction between the contorsion tensor (13.33) and the spin density (13.27):

13.35

$$\mathcal{L}_{\mathrm{D}} = \overset{\circ}{\mathcal{L}}_{\mathrm{D}} + \frac{1}{4} i C_{abc} \bar{\psi} \left\{\gamma^a, \sigma^{bc}\right\} \psi = \overset{\circ}{\mathcal{L}}_{\mathrm{D}} + \frac{1}{2} C_{abc} \sigma^{abc}$$

□ The spin tensor of the Dirac field is completely antisymmetric. This implies that the torsion tensor, algebraically related to the spin density through the Cartan equations (13.5), and the contorsion tensor, are both *completely anti-symmetric* as well. Thus, in the case of the Dirac field, one has the simple relationship $2C^{abc} = T^{abc} = \kappa\sigma^{abc}$. It concisely summarizes that in ECKS gravity the torsional degrees of freedom solely originate from the presence of fermions.

□ When the linear relationship is inserted into equation (13.35), it induces an effective *spin-spin interaction* in the Lagrangian:

13.36

$$\mathcal{L}_{\mathrm{D}} = \overset{\circ}{\mathcal{L}}_{\mathrm{D}} + \frac{1}{4}\kappa\sigma_{abc}\sigma^{abc} = \overset{\circ}{\mathcal{L}}_{\mathrm{D}} - \frac{3}{8}\kappa\left(\bar{\psi}\gamma_d\gamma^5\psi\right)\bar{\psi}\gamma^d\gamma^5\psi$$

In the last term the spin density tensor has been expressed in terms of the Dirac *axial spin vector*, see (D.3):

13.37

$$\sigma^{abc} = \varepsilon^{abcd} j_d \quad j_d := \frac{1}{2}\bar{\psi}\gamma_d\gamma^5\psi$$

The torsion due to the presence of fermions is now effectively contained in the matter sector in the form of an 4-spinor interaction term embedded in a *Riemannian geometry*, i.e. standard GRT.

□ Variation of the corresponding action with respect to the spinor field $\bar{\psi}$ yields the nonlinear *Hehl-Datta equation*:

13.38

$$i\hbar\gamma^a \mathring{\mathcal{D}}_a\psi = m\psi + \frac{3\kappa\hbar^2}{8}(\bar{\psi}\gamma_d\gamma^5\psi)\gamma^d\gamma^5\psi$$

The Planck constant $\hbar$ is made explicit to show that the coupling constant in this effective equation is proportional to square of the Planck length $l_\mathrm{P} := \sqrt{\hbar G/c^3}$. This indicates that deviations of GRT are only likely to occur in extreme density regimes. [Wikipedia: Nonlinear Dirac Equation]

Appendix A

Levi-Civita Tensor [MTW,CC]

☐ In the tetrad formalism, it is natural to introduce the completely anti-symmetric *Levi-Civita symbol* ε_{abcd}, invariant under $\mathrm{SO}(1,3)$, with values ± 1. Its normalization is chosen to be $\varepsilon_{0123} = 1 = -\varepsilon^{0123}$, i.e. the Levi-Civita symbol equals the sign of a permutation when the indices are all unequal. De minus sign reflects the metric signature of the Lorentzian manifold.

☐ By means of the Levi-Civita symbol, the unit volume element and its inverse, defined in Volume Element, may be written according to

A.1

$$\mathrm{I}_4 = -\frac{\varepsilon^{abcd}}{4!}\mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c \wedge \mathbf{e}_d \qquad \mathrm{I}^4 = \frac{\varepsilon_{abcd}}{4!}\mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d$$

These relations may be inverted to give following useful representations of the unit 4-blades:

A.2

$$\mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c \wedge \mathbf{e}_d = \varepsilon_{abcd}\,\mathrm{I}_4 \qquad \mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d = -\varepsilon^{abcd}\,\mathrm{I}^4$$

☐ The name 'symbol' emphasizes that the Levi-Civita symbol does not behave as a tensor: under a general coordinate change, the components of the permutation tensor are multiplied by the Jacobian of the transformation matrix. Objects that transform that way are known as *tensor densities*. [Wikipedia: Levi-Civita symbol]

☐ In curved spacetime, one may construct a covariant *Levi-Civita tensor* (also known as the Riemannian volume form) by defining

A.3

$$\epsilon_{\mu\nu\kappa\lambda} := \sqrt{|g|}\varepsilon_{\mu\nu\kappa\lambda} \qquad \epsilon^{\mu\nu\kappa\lambda} := |g|^{-1}\epsilon_{\mu\nu\kappa\lambda} = |g|^{-1/2}\varepsilon^{\mu\nu\kappa\lambda}$$

These formulae are consistent with the vierbein transformation

A.4

$$\epsilon_{\mu\nu\kappa\lambda} \;= e_\mu{}^a e_\nu{}^b e_\kappa{}^c e_\lambda{}^d \varepsilon_{abcd} = \det[e_\mu{}^a]\varepsilon_{\mu\nu\kappa\lambda} = \sqrt{|g|}\varepsilon_{\mu\nu\kappa\lambda}$$

and similarly for the second equation in (A.3).

☐ The above formulae may be extended to spaces of arbitrary dimension n with generalized orthonormal metric $\mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab} = \mathrm{diag}(1, ..., 1, -1, ..., -1)$ of Lorentz signature $(n-s, s)$, where s is the number of negative elements. Indices are raised and lowered with the generalized metric; in particular $\varepsilon^{0123...n-1} = (-1)^s \varepsilon_{012...n-1}$.

Appendix B

Generalized Kronecker Delta [CC]

☐ In calculations with the Levi-Civita symbol (tensor), it is often convenient to express the result in terms of a so-called *generalized Kronecker delta* defined by the determinant:

B.1

$$\delta^{a_1 a_2 \ldots a_p}_{b_1 b_2 \ldots . b_p} := p! \delta^{a_1}_{[b_1} \ldots \delta^{a_p}_{b_p]} = \begin{vmatrix} \delta^{a_1}_{b_1} & .. & \delta^{a_1}_{b_p} \\ .. & & .. \\ \delta^{a_p}_{b_1} & .. & \delta^{a_p}_{b_p} \end{vmatrix}$$

as a completely anti-symmetric unity tensor. [Wikipedia: Kronecker delta]

☐ For example, when $p = n$ (the dimension of the vector space), one may derive the compact formula

B.2

$$\varepsilon^{a_1 \ldots a_n} \varepsilon_{b_1 \ldots b_n} = (-1)^s \delta^{a_1 \ldots a_n}_{b_1 \ldots . b_n}$$

More generally

B.3

$$\frac{1}{(n-p)!} \varepsilon^{a_1 \ldots a_p c_{p+1} \ldots c_n} \varepsilon_{b_1 \ldots b_p c_{p+1} \ldots c_n} = (-1)^s \delta^{a_1 \ldots a_p}_{b_1 \ldots b_p}$$

The formulae for $p = 1, 2, 3,\ n = 4$

B.4

$$\begin{aligned} \tfrac{1}{3!} \varepsilon_{abcd} \varepsilon^{abck} &= (-1)^s \delta^k_d \\ \tfrac{1}{2!} \varepsilon_{abcd} \varepsilon^{abkl} &= (-1)^s \delta^{kl}_{cd} \\ \tfrac{1}{1!} \varepsilon_{abcd} \varepsilon^{aklm} &= (-1)^s \delta^{klm}_{bcd} \end{aligned}$$

prove to be particularly useful.

Appendix C

Duality Operation [AT, DL,EC]

☐ The forming of the product of the grade-n (pseudoscalar) volume element I_n of a geometric vector space $\mathcal{G}_n$, with a grade-p multivector $\mathbf{A}_p$, is called a *duality operation*:

C.1

$$\mathbf{A}_p \to *\mathbf{A}_p := \mathrm{I}_n \mathbf{A}_p$$

If $\mathbf{A}_p$ is a blade, $\mathrm{I}_n\mathbf{A}_p$ returns a blade of grade $r = n - p \leqslant n$, embedded in the subspace of vectors not contained in $\mathbf{A}_p$. This blade $*\mathbf{A}_p$ is called the *dual* of $\mathbf{A}_p$ or its *orthogonal complement*. The operation can be reversed by multiplication with the inverse I^n. Both are duality operations equivalent (up to a sign) to the *Hodge Star operation* in the theory of differential forms.

☐ Since the duality operation is linear, it is sufficient to consider how the volume element I_n acts on orthonormal p-blades. For $1 \leqslant p \leqslant n$ it follows with (12.3):

C.2

$$\mathrm{I}_n \mathbf{e}^{a_1} \wedge ... \wedge \mathbf{e}^{a_p} = (-1)^s \frac{(-1)^{r(r-1)/2}}{r!} \varepsilon^{a_1 .... a_p a_{p+1} ... a_n} \mathbf{e}_{a_{p+1}} \wedge ... \wedge \mathbf{e}_{a_n}$$

The sign depends on the number of permutations that must be performed to bring both sets of indices $\{p+1, ..., n\}$ at the right-hand side in ascending numerical order.

☐ Given a geometric vector space $\mathcal{G}_n$ with $\{\mathbf{e}_i; i = 1, .., n\}$ an orthonormal base. The set of all possible blades $\{1, \mathbf{e}_i, \mathbf{e}_i \wedge \mathbf{e}_j, \mathbf{e}_i \wedge \mathbf{e}_j \wedge \mathbf{e}_k, ...\}$ with $i < j < k < ..$, provides a basis for the entire algebra which has the dimension $\dim(\mathcal{G}_n) = 2^n$. The fact that the basis vectors anti-commute ensures that each product in the basis set is totally antisymmetric.

☐ In particular, the space $\mathcal{G}_4$ has $2^4 = 16$ linearly independent elements: one scalar, four vectors, six bivectors, four trivectors and one speudoscalar. Applying the duality operation $\{\mathbf{e}\} \to \mathrm{I}^4\{\mathbf{e}\}$, one derives:

C.3 Trautman Forms

a. $\eta := \mathrm{I}^4$

b. $\boldsymbol{\eta}_a := \mathrm{I}^4\mathbf{e}_a = \frac{1}{3}\mathbf{e}^b \wedge \boldsymbol{\eta}_{ab} = -\frac{1}{3!}\varepsilon_{abcd}\mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d$

c. $\boldsymbol{\eta}_a := \mathrm{I}^4\mathbf{e}_a = \frac{1}{3}\mathbf{e}^b \wedge \boldsymbol{\eta}_{ab} = -\frac{1}{3!}\varepsilon_{abcd}\mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d$

d. $\boldsymbol{\eta}_{abc} := \mathrm{I}^4\, \mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c = \varepsilon_{abcd}\mathbf{e}^d$

e. $\eta_{abcd} := \mathrm{I}^4\, \mathbf{e}_a \wedge \mathbf{e}_b \wedge \mathbf{e}_c \wedge \mathbf{e}_d = \varepsilon_{abcd}$

These $\boldsymbol{\eta}$-forms, first introduced by Andrzej Trautman (1973), constitute a *dual basis* for the algebra of the geometric space $\mathcal{G}_4$.

Appendix D

Dirac Algebra [CK]

☐ The *Dirac Algebra* is a complex matrix representation of the Clifford Algebra Cl$(1,3)$ generated by a set of 4x4 matrices $\{\gamma^a, \gamma_a; a = 0,1,2,3\}$. The *gamma matrices* formally constitute an orthonormal basis of a (local) Minkowski spacetime defined by the inner product (anticommutator) relation:

D.1

$$\gamma^a \cdot \gamma^b := \frac{1}{2}\left\{\gamma^a, \gamma^b\right\} = \eta^{ab} I_4$$

where η^{ab} is the Minkowski metric with signature $(+,-,-,-)$. It is conventional to suppress component indices of the gamma matrices and spinors. Usually, the identity matrix at the right-hand side is also omitted. The wedge product (commutator)

D.2

$$\gamma^a \wedge \gamma^b := \frac{1}{2}\left[\gamma^a, \gamma^b\right] := 2\sigma^{ab}$$

defines the *spin matrices* σ^{ab}. These form the spin representation of the Lorentz group, so that $S(\Lambda) = \exp\left(\frac{1}{2}\varepsilon_{ab}\sigma^{ab}\right)$ are Lorentz transformations in spinor space; see Gauge Theory.

☐ There are many matrix representations that conform to the anticommutator rule (D.1), amongst which the Standard (Dirac-Pauli) representation and the Chiral (Weyl) representation. [Wikipedia: Gamma Matrices]

D.3 Common Properties

a. $\left(\gamma^0\right)^2 = \left(\gamma_0\right)^2 = 1, \quad \left(\gamma^0\right)^\dagger = \gamma^0$

b. $\left(\gamma^i\right)^2 = \left(\gamma_i\right)^2 = -1, \ \left(\gamma^i\right)^\dagger = -\gamma^i, \quad i = 1,2,3$

c. $\left(\gamma^a\right)^\dagger = \gamma^0\gamma^a\gamma^0, \quad \left(\sigma^{ab}\right)^\dagger = -\gamma^0\sigma^{ab}\gamma^0, \quad \left(\gamma^0\gamma^a\right)^\dagger = \gamma^0\gamma^a$

d. $\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3, \quad \gamma_5 := -i\gamma_0\gamma_1\gamma_2\gamma_3, \quad \left(\gamma^5\right)^2 = 1$

e. $(\gamma^5)^\dagger = \gamma^5 = \gamma_5 = (\gamma_5)^{-1}$, $(\gamma^5\gamma^a)^\dagger = \gamma^0\gamma^5\gamma^a\gamma^0$

The dagger operator implements the complex conjugation of the transpose.

□ The matrix representations of the unit pseudo-scalar elements of the Dirac Algebra are conventionally denoted as the 'fifth' matrices $\{\gamma^5, \gamma_5\}$; they have an imaginary unit included in their definition so as to have real eigenvalues. These pseudo scalars anti-commute with the four gamma matrices $\{\gamma^5, \gamma^a\} = 0$ and commute with the spin matrices: $\left[\gamma^5, \sigma^{ab}\right] = 0$.

Appendix E

Tetrad Algebra [DH,DL,FH]

☐ The rules for calculating the *inner product* of a vector with a p-form and the *inner product* of two p-forms are:

E.1

$$\mathbf{a}\cdot(\mathbf{a}_1\wedge...\wedge\mathbf{a}_p)=\sum_{k=1}^{p}(-1)^{k+1}(\mathbf{a}\cdot\mathbf{a}_k)\mathbf{a}_1\wedge...\wedge\breve{\mathbf{a}}_k\wedge...\wedge\mathbf{a}_p$$

E.2

$$(\mathbf{a}_p\wedge...\wedge\mathbf{a}_1)\cdot(\mathbf{b}_1\wedge...\wedge\mathbf{b}_p)=\begin{vmatrix}\mathbf{a}_1\cdot\mathbf{b}_1 & .. & \mathbf{a}_1\cdot\mathbf{b}_p\\ .. & & ..\\ \mathbf{a}_p\cdot\mathbf{b}_1 & .. & \mathbf{a}_p\cdot\mathbf{b}_p\end{vmatrix}$$

The 'check' on $\mathbf{a}_k$ in (E.1) indicates omission. Note the *reverse order* of the **a**-blade in (E.2). Otherwise, a sign-factor $(-1)^{p(p-1)/2}$ needs to be factored in because it takes $p(p-1)/2$ interchanges to put the **a**-vectors in *opposite* order. This operation of *reversion* on a blade $\mathbf{A}_p=\mathbf{a}_1\wedge...\wedge\mathbf{a}_p$ is indicated by the tilde-symbol: $\widetilde{\mathbf{A}}_p:=\mathbf{a}_p\wedge...\wedge\mathbf{a}_1$.

☐ Reversion is included in the definition of the *scalar product* of two multivectors that yields the scalar part:

E.3

$$\langle\mathbf{A}_p,\mathbf{B}_p\rangle:=\widetilde{\mathbf{A}}_p\cdot\mathbf{B}_p$$

The scalar product is non-zero only when $\mathbf{A}_p$ and $\mathbf{B}_p$ are of the same grade. It is symmetrical and reversable, and may be used to define a 'norm squared' of blades and multivectors, e.g.: $\|\mathbf{A}_p\|^2:=\left|\widetilde{\mathbf{A}}_p\cdot\mathbf{A}_p\right|=|\mathrm{Det}\,[\mathbf{a}_i\cdot\mathbf{a}_j]|$.

☐ Identities for wedge products involving Trautman forms:

E.4

a.	$\mathbf{e}^k \wedge \boldsymbol{\eta}_a \quad = -\delta^k_a \eta, \quad \eta := \mathrm{I}^4$
b.	$\mathbf{e}^k \wedge \boldsymbol{\eta}_{ab} \quad = -2!\delta^k_{[a}\boldsymbol{\eta}_{b]}$
c.	$\mathbf{e}^k \wedge \boldsymbol{\eta}_{abc} = -3!\delta^k_{[a}\boldsymbol{\eta}_{bc]}$
d.	$\mathbf{e}^k \wedge \varepsilon_{abcd} = -4!\delta^k_{[a}\boldsymbol{\eta}_{bcd]}$
e.	$\mathbf{e}^k \wedge \mathbf{e}^l \wedge \boldsymbol{\eta}_{ab} \quad = -2!\delta^k_{[a}\delta^l_{b]}\eta$
f.	$\mathbf{e}^k \wedge \mathbf{e}^l \wedge \boldsymbol{\eta}_{abc} = -3!\delta^k_{[a}\delta^l_b\boldsymbol{\eta}_{c]}$
g.	$\mathbf{e}^k \wedge \mathbf{e}^l \varepsilon_{abcd} \quad = -4!\delta^k_{[a}\delta^l_b\boldsymbol{\eta}_{cd]}$

To obtain these identities one identifies the p-blades at the left-hand sides, takes the dual with (C.2) and applies (B.3). Then the inverse duality operation gives the Trautman forms at the right-hand sides.