LECTURE NOTES

Statistical Field Theory

An Introduction to Real- and Imaginary-time Thermal Field Theory

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2001

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Chapter 1

Introduction

In these lecture notes we discuss some aspects of quantum fields at finite temperature and chemical potential. We will use the name "Thermal Field Theory (TFT)" to indicate the modern version of quantum-mechanical many-body theory originally developed in the fifties and sixties. TFT deals with the behavior of large assemblies of elementary particles at non-zero temperature, and can be considered as an amalgam of statistical mechanics and elementary-particle physics with applications to the early universe, to astrophysics and to ultra-relativistic nucleus-nucleus collisions.

We shall be concerned with the basic formal content of TFT and its perturbative structure. In the first chapters we shall mainly look at the oldest and most often used imaginary-time (Euclidean) formalism. It is based on the formal analogy, first noted by F. Bloch in 1932, between inverse temperature and imaginary time. It leads to so-called temperature Green functions with purely imaginary time arguments. That is, one works in Euclidean space. The formalism was developed by many authors but is usually named after Matsubara (1955), who was the first to set up a diagrammatic perturbation theory for the grand partition function on a field-theoretic basis.

1.1 Some history

Before we introduce the basic concepts of Thermal Field Theory we start with a short historical survey. In the fifties several physicists, among them Kadanoff, Martin and Schwinger, realized that field-theoretic methods were useful in the study of non-relativistic quantum problems as well. It was discovered that practically all statistical observables in, or close to thermal equilibrium, like the equation of state, the thermodynamic response functions and most transport coefficients, could be expressed in terms of Green functions similar to those encountered in quantum field theory. This led to the development of perturbative techniques based on Feynman diagrams, as well as more general methods based on the use of the exact Green functions. This Many-Body Theory, as it was called was codified in a number of well-known text books, e.g. Fetter and Walecka (1971).

The generality of the results invited the extension to relativistic problems by Silin and Fradkin in the sixties. In the early seventies new developments were triggered by

the interest in the problem of symmetry restoration at high temperature. This new direction in thermal field theory came about through a paper by Kirzhnits and Linde in 1972, who pointed out that one should expect a symmetry-restoring phase transition in the Weinberg-Salam model of weak interaction, in analogy to ferromagnetism and superconductivity. The general idea that broken symmetries in relativistic quantum field theory should be restored at high temperature was subsequently elaborated by Weinberg (1974), Dolan and Jackiw (1974) and by Linde himself.

The obvious next step was to study Quantum Chromo Dynamics (QCD) in thermal field theory. This was initiated by Collins and Perry (1975), who argued that the strong interaction becomes weak not only at very high momentum transfer, as in deep inelastic scattering, but in ultra-dense nuclear matter as well. This seems evident from the usual heuristic picture of asymptotic freedom. The argument was rapidly extended to the case of low densities but very high temperatures (~ 150 MeV). The idea of asymptotic freedom at high temperature and/or density led to the notion of the Quark-Gluon Plasma (a name coined by Shuryak (1980). In the most naive picture, the phase diagram of QCD has two regions: a low-temperature phase in which quarks and gluons are confined, and a high-temperature phase (the Quark-Gluon Plasma) in which these particles are liberated and interact only weakly. The existence of this new state of matter may have been experimentally confirmed at CERN last year.

In recent years thermal field theory also has been applied with some success to other high-temperature phase transitions in gauge theories. For example, a first-order phase transition in the standard electroweak theory at $T \sim 100 \text{GeV} = 10^{15}\text{K}$, about $t = 10^{-10}$ seconds after the Big Bang, may drive baryogenesis. Below the electroweak phase transition, the particles in the standard model acquire their masses through the Higgs mechanism. Above the phase transition, the Higgs expectation value is zero and the particles are massless. To determine the critical temperature, the effective potential at high-temperature has to be calculated. An idea that has been elaborated by Kajantie and others (1994) is to calculate the effective potential in a dimensional reduction scheme by integrating out all modes, except the lightest. This allows one to consider 4D field theory at finite temperature as a 3D field theory at zero temperature, with an effective mass and coupling constant.

1.2   Grand canonical ensemble

The state of a system of many particles in equilibrium is characterized by the conserved quantities of the system. For a quantum system these usually are the time-independent Hamiltonian $\hat{H}$, and a number of conserved and mutually commuting charges $\hat{Q}_a, a = 1, 2, ...$, such as lepton number, baryon number, etc. (The circumflex indicates a quantum mechanical operator in the Heisenberg picture.) In statistical mechanics it is assumed that these quantities give a complete macroscopic description of the equilibrium state for a system at rest in a large volume V.

Because in relativistic field theory particles are continually being created and de-

stroyed, it is appropriate to use the grand canonical ensemble in which the average charge densities N_a and the energy density E are fixed. By maximizing the entropy functional

$$S[\rho] = -\frac{1}{V}\mathrm{Tr}\hat{\rho}\log\hat{\rho} \tag{1.1}$$

for the density operator $\hat{\rho}$, one finds that the most probable configuration is specified by the grand canonical density operator

$$\hat{\rho} = \exp\left(V\Omega - \sum_a \alpha_a \hat{Q}_a - \beta\hat{H}\right) . \tag{1.2}$$

Since any density operator is by definition a positive operator of unit trace: $\mathrm{Tr}\hat{\rho} = 1$, the thermodynamic potential $V\Omega = -\log Z$ is found by calculating the grand canonical partition function

$$Z(\alpha_a, \beta, V) = \mathrm{Tr}\exp\left(-\sum_a \alpha_a \hat{Q}_a - \beta\hat{H}\right) , \tag{1.3}$$

where the trace may be taken over any complete (not necessarily orthonormal) set of states. The Lagrange multipliers α_a, β are related to the temperature T and the independent chemical potentials μ_a through $\beta^{-1} = k_B T, \alpha_a = -\beta\mu_a$, and are conjugate to the charge densities and energy density through the relations

$$N_a = \frac{1}{V}\langle\hat{Q}_a\rangle = \frac{\partial\Omega}{\partial\alpha_a} , \tag{1.4}$$

$$E = \frac{1}{V}\langle\hat{H}\rangle = \frac{\partial\Omega}{\partial\beta} , \tag{1.5}$$

where the brackets indicate the grand-canonical average. For some systems of bosons there are no conserved quantum numbers $\hat{Q}$ and therefore no corresponding Lagrange multiplier. Then the grand-canonical ensemble reduces to the canonical ensemble $\exp -\beta\hat{H}$. [1]

From the thermodynamic potential Ω, we may define

$$P(T, \mu_a) = -\frac{1}{\beta}\Omega , \tag{1.6}$$

which may be identified with the thermodynamic pressure on account of the variational relation

$$\delta P = S\delta T + \sum_a N_a \delta\mu_a , \tag{1.7}$$

which is an immediate consequence of the above formulae. This shows that the thermodynamic pressure, as a function of the independent variables T and μ_a, is a characteristic thermodynamic function from which other thermodynamic quantities may be obtained by

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[1]In the following we will use units such that $k_B = 1, \hbar = 1, c = 1$; but where appropriate these constants will be written explicitly.

differentiation. The additivity of the entropy $SV = -\langle \log \hat{\rho} \rangle$ provides us with the Euler relation

$$S = \beta P + \beta E + \sum_a \alpha_a N_a, \qquad (1.8)$$

which expresses the entropy density S in terms of the other thermodynamic variables.

Since many applications of thermal field theory concern relativistic systems one might well ask, how the Gibbs ensembles are adaptable to such situations. The answer is that thermodynamics and statistical mechanics can be formulated in a covariant way by introducing the notion of a local observer moving at some hydrodynamic velocity $U^\mu(x), U^\mu U_\mu = 1$. For a system in equilibrium it is sufficient to consider a global hydrodynamic velocity characterizing the system, which has the components $U^\mu = (1, 0, 0, 0)$ for a system at rest. The maximum entropy principle can be applied in the standard manner and leads to the grand canonical ensemble of the form (1.2) with the only difference that the Hamiltonian and the charges are now defined with respect to the rest frame

$$\hat{Q}_a = U_\mu \hat{J}_a^\mu , \qquad (1.9)$$
$$\hat{H} = U_\mu \hat{P}^\mu , \qquad (1.10)$$

where $\hat{J}_a$ and $\hat{P}^\mu$ are the conserved currents and energy momentum operators, respectively, of the system. The temperature T, the chemical potentials μ_a, and all other thermodynamic quantities, are always measured with respect to the rest frame.

Thermodynamic relations can be written in a covariant form by introducing the macroscopic current flows and energy-momentum tensor

$$J_a^\mu = \langle \hat{J}_a^\mu \rangle = N_a U^\mu , \qquad (1.11)$$
$$T^{\mu\nu} = \langle \hat{T}^{\mu\nu} \rangle = (E + P)U^\mu U^\nu - g^{\mu\nu} P . \qquad (1.12)$$

In the rest frame, the energy-momentum tensor is diagonal with diagonal components $T^{\mu\nu} = \text{Diag}(E, P, P, P)$. In terms of these covariant quantities we may write the entropy flow as

$$S^\mu = SU^\mu = \alpha_a J_a^\mu + \beta_\mu T^{\mu\nu} - \Omega U^\mu , \qquad (1.13)$$

where the inverse temperature four-vector has the definition $\beta_\mu = \beta U_\mu$. It follows from these equations that

$$\delta(P\beta^\mu) = J_a^\mu \delta\alpha_a + T^{\nu\mu} \delta\beta_\nu . \qquad (1.14)$$

The last two relations show how the basic variables $J^\mu, T^{\mu\nu}, S^\mu$, can all be generated from partial derivation of the thermodynamic four-potential $\Omega^\mu = -P\beta^\mu$.

In the covariant formulation, the hydrodynamic velocity becomes a thermodynamic variable on par with the temperature and the chemical potentials. Since the covariant thermodynamic relations are expressed directly in terms of the basic conserved currents and energy-momentum tensor, they are more transparent. Physically, however, no new information is gained, and in the rest frame the standard thermodynamic relationships are recovered.

1.3 Scalar Bose field

The thermal average of an observable corresponding to some quantum mechanical operator $\hat{A}$ is defined as

$$\langle\hat{A}\rangle = \frac{1}{Z}\text{Tr}\hat{A} \ . \tag{1.15}$$

In particular we will be interested in the thermal average of a number of fields. Consider, for example, the case of a Hermitian scalar Bose field $\hat{\phi}(x)$, carrying no conserved charges, with conjugate momentum $\hat{\pi}(x) = \partial_0\hat{\phi}(x)$, whose dynamics is governed by the Hamiltonian

$$\hat{H} = \int d^3x \ \left[\tfrac{1}{2}\hat{\pi}^2 + \tfrac{1}{2}(\nabla\hat{\phi})^2 + \mathcal{V}(\hat{\phi})\right] \ . \tag{1.16}$$

We define

$$\hat{\phi}(x) = e^{it\hat{H}/\hbar}\hat{\phi}(0,\mathbf{x})e^{-it\hat{H}/\hbar} \ , \tag{1.17}$$

where the time coordinate $x^0 = ct$ may be complex. Observable dynamical properties of the system can be extracted from thermal correlation functions

$$C(x_1, x_2, \ldots, x_N) = \langle\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_N)\rangle \tag{1.18}$$

defined as the statistical average of a product of Heisenberg fields. In equilibrium these correlation functions are translation invariant, meaning that they only depend on the coordinate differences. Moreover equilibrium correlation functions satisfy the so called KMS (Kubo-Martin-Schwinger) condition:

$$\langle\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_N)\rangle = \langle\hat{\phi}(x_2)\ldots\hat{\phi}(x_N)\hat{\phi}(x_1 + i\hbar\beta)\rangle \ . \tag{1.19}$$

This is a special case of a fundamental equivalence between quantum field theory and quantum statistical mechanics, based on the formal correspondence between imaginary time and inverse temperature:

$$e^{-\beta\hat{H}} \leftrightarrow e^{i\hat{H}t/\hbar}, \quad t = i\beta\hbar \ , \tag{1.20}$$

which was first noted by Felix Bloch in 1932. The proof of (1.19) involves no more than the cyclic invariance of the trace.

Let us calculate the 2-point correlation function for the system without interaction. The field may be expanded in the usual manner

$$\hat{\phi}(x) = \sum_k \left[f_k(x)\hat{a}_k + f_k^*(x)\hat{a}_k^\dagger\right] \tag{1.21}$$

in terms of creation and annihilation operators satisfying the commutation relations:

$$\left[\hat{a}_k^\dagger, \hat{a}_l^\dagger\right] = 0 \ , \tag{1.22}$$

$$\left[\hat{a}_k, \hat{a}_l\right] = 0 \ , \tag{1.23}$$

$$\left[\hat{a}_k, \hat{a}_l^\dagger\right] = \delta_{kl} \ . \tag{1.24}$$

The labels k, l stand for all quantum numbers needed to specify the complete set of mode functions satisfying the Klein-Gordon equation $(\partial^2 + m^2)f_k(x) = 0$. For spinless bosons in a finite volume $V = L^3$, these are simply plane waves

$$f_k(x) = \sqrt{\frac{\hbar c}{2\omega_k V}} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega_k t} ,\tag{1.25}$$

with $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$ and momentum quantum numbers $k = 2\pi\mathbf{n}/L, \mathbf{n} = (n_1, n_2, n_3)$, n_i integer. The mode functions are normalized by the Wronskian-type conditions

$$\sum_k [f_k(x)\partial_t f_k^*(x')_{t=t'} - f_k^*(x)\partial_t f_k(x')_{t=t'}] = i\hbar c\, \delta(\mathbf{x} - \mathbf{x}') ,\tag{1.26}$$

$$\int d^3x\ [f_k(x)\partial_t f_l^*(x')_{t=t'} - f_k^*(x)\partial_t f_l(x')_{t=t'}] = i\hbar c\, \delta_{kl} ,\tag{1.27}$$

as required by the canonical equal-time commutation relation

$$\left[\hat{\phi}(x), \hat{\pi}(x')\right]_{t=t'} = i\hbar\, \delta(\mathbf{x} - \mathbf{x}') .\tag{1.28}$$

Before closing this section we make a short remark about dimensions. Usually we will set $\hbar = c = 1$, so that energy will have the dimension of inverse length: $[E] = [L^{-1}]$. The constants $c, \hbar$ can always be restored by dimensional analysis. Suppose now that we set $c = 1$ but keep $\hbar$. This is useful for identifying classical contributions. Since $\hbar$ has a dimension of energy times length, $[\hbar] = [EL]$, we no longer have $[E] = [L^{-1}]$. By inspection of the Hamiltonian, we find then the following dimensions: $[\phi^2] = [EL^{-1}], [m] = [L^{-2}], [k] = [\omega_k] = [L^{-1}]$.

1.4 Ideal Bose gas

Because the operators $\hat{a}_k, \hat{a}_k^\dagger$ destroy or create modes with energy $\epsilon_k = \hbar\omega_k$, the free Hamiltonian reads

$$\hat{H}_0 = \tfrac{1}{2}\sum_k \epsilon_k(\hat{a}_k^\dagger\hat{a}_k + \hat{a}_k\hat{a}_k^\dagger) ,\tag{1.29}$$

which may be checked by substitution of the mode decomposition of the field into (1.16). It is now easy to compute the statistical average

$$\langle\hat{a}_k^\dagger\hat{a}_l\rangle_0 = \frac{1}{Z_0}\mathrm{Tr}\, e^{-\beta\hat{H}_0}\hat{a}_k^\dagger\hat{a}_l\tag{1.30}$$

for a free system without actually performing the trace. The cyclic invariance of the trace allows us to write

$$\langle\hat{a}_k^\dagger\hat{a}_l\rangle_0 = \langle\hat{a}_l\hat{a}_k^\dagger(i\beta\hbar)\rangle_0 ,\tag{1.31}$$

where we used the abbreviation:

$$\hat{a}_k^\dagger(i\beta\hbar) = e^{-\beta\hat{H}_0}\hat{a}_k^\dagger e^{\beta\hat{H}_0} .\tag{1.32}$$

In consequence of the simple commutation relation

$$\left[\hat{H}_0, \hat{a}_k^\dagger\right] = \epsilon_k \hat{a}_k^\dagger , \tag{1.33}$$

and the general rule

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = B + \left[\hat{A}, \hat{B}\right] + \tfrac{1}{2}\left[\hat{A}, \left[\hat{A}, \hat{B}\right]\right] + \cdots , \tag{1.34}$$

we obtain

$$\hat{a}_k^\dagger(i\beta\hbar) = \hat{a}_k^\dagger e^{-\beta\epsilon_k} . \tag{1.35}$$

We substitute this result back into (1.31) and use the commutation relation (1.24). Solving for the desired thermal average we get

$$\langle \hat{a}_k^\dagger \hat{a}_l \rangle_0 = \delta_{kl}\, n(\epsilon_k), \tag{1.36}$$

where

$$n(\epsilon) = \frac{1}{e^{\beta\epsilon} - 1} \tag{1.37}$$

is the Bose-Einstein distribution function. Therefore, in equilibrium the expectation value (1.30) is diagonal and can be interpreted as the probability for finding a particle with energy ϵ_k. Similarly, we obtain

$$\langle \hat{a}_k \hat{a}_l^\dagger \rangle_0 = \delta_{kl}\left[1 + n(\epsilon_k)\right] = -\delta_{kl}\, n(-\epsilon_k) . \tag{1.38}$$

The thermal averages of two creation or two annihilation operators vanish by the same reasoning. Moreover the average of any odd number of operators vanishes.

Problem 1.1

  a. *Show that*

$$\langle \hat{a}_k \hat{a}_l \rangle_0 = 0 \tag{1.39}$$

  and that the average of an odd number of operators vanishes.

  b. *Extend the reasoning to obtain*

$$\langle \hat{a}_{k_1}^\dagger \ldots \hat{a}_{k_n}^\dagger \hat{a}_{l_n} \ldots \hat{a}_{l_1} \rangle_0 = \sum_{j=1}^n \langle \hat{a}_{k_1}^\dagger \hat{a}_j \rangle_0 \langle \hat{a}_{k_2}^\dagger \ldots \hat{a}_{k_n}^\dagger \hat{a}_{l_n} \ldots \hat{\not{a}}_j \ldots \hat{a}_{l_1} \rangle_0 \tag{1.40}$$

  and show by induction that any expectation values factorizes into averages of pairs of creation and annihilation operators. This is called the thermodynamic Wick theorem.

The above results may be used to calculate the partition function of the free neutral boson gas

$$Z = \mathrm{Tr}\exp -\tfrac{1}{2}\beta \sum_k \epsilon_k(\hat{a}_k^\dagger \hat{a}_k + \hat{a}_k \hat{a}_k^\dagger) \tag{1.41}$$

by first calculating the energy density

$$E = -\frac{1}{V}\frac{\partial \log Z}{\partial \beta} = \frac{1}{V}\sum_k [\epsilon_k n(\epsilon_k) + \tfrac{1}{2}\epsilon_k] = \frac{1}{2V}\sum_k \epsilon_k \frac{e^{\frac{1}{2}\beta\epsilon_k} + e^{-\frac{1}{2}\beta\epsilon_k}}{e^{\frac{1}{2}\beta\epsilon_k} - e^{-\frac{1}{2}\beta\epsilon_k}} \ . \tag{1.42}$$

The result includes the zero-point energy of the vacuum, which one may wish to subtract off since it is a temperature independent (infinite) constant. Integrating with respect to the temperature we get for the thermodynamic potential

$$\Omega = \frac{1}{V}\sum_k \left[\log(1 - e^{-\beta\epsilon_k}) + \tfrac{1}{2}\beta\epsilon_k\right] \ . \tag{1.43}$$

In the thermodynamic limit we are allowed to make the replacement from sum to integral by the substitution:

$$\frac{1}{V}\sum_k \to \int \frac{d^3k}{(2\pi)^3} \ . \tag{1.44}$$

Then we get

$$\Omega = \int \frac{d^3k}{(2\pi)^3} \left[\log(1 - e^{-\beta\epsilon_k}) + \tfrac{1}{2}\beta\epsilon_k\right] \ . \tag{1.45}$$

This is the well known expression for the thermodynamic potential of an ideal Bose gas when the number of particles is not conserved.

Let us now to calculate the 2-point correlation function for a free system and the associated spectral density, that is, the commutator of the fields. We anticipate that these correlation functions will only depend on the relative coordinate, on account of the translational invariance of thermal averages. Introducing the four-vector $k^\mu = (k_0, \mathbf{k})$, we write

$$\langle \hat{\phi}(x)\hat{\phi}(x')\rangle_0 = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}\tilde{C}_0^>(k) \ , \tag{1.46}$$

$$\langle [\hat{\phi}(x), \hat{\phi}(x')]\rangle_0 = \hbar c \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-x')}\rho_0(k) \ , \tag{1.47}$$

where $\rho_0(k)$ is the spectral density. Going to the continuum limit we find by explicit calculation:

$$\tilde{C}_0^<(k) = \hbar c \rho_0(k)[1 + n(k_0)] \ , \tag{1.48}$$

$$\rho_0(k) = 2\pi \mathrm{sign}(k_0)\delta(k^2 - m^2) = \frac{\pi}{\omega_k}[\delta(k_0 - \omega_k) - \delta(k_0 + \omega_k)] \ . \tag{1.49}$$

The right-hand side in the last line is obtained with the help of the delta-function identity

$$\delta\left(f(x) - f(x_0)\right) = \frac{1}{|f'(x_0)|}\delta(x - x_0) \ . \tag{1.50}$$

Note that the energy k_0 in the expression (1.48) is an independent variable, not the mass shell energy ω_k.

Problem 1.2

a. *Show that the spectral density is an odd function of the energy with the positivity property*

$$k_0 \rho_0(k) \geq 0 , \qquad (1.51)$$

b. *and has the normalization*

$$\int_0^\infty \frac{dk_0^2}{2\pi} \rho_0(k) = 1 . \qquad (1.52)$$

Problem 1.3

Show that the Fourier transform of the correlation function $C_0^<(x,x') \equiv bra\hat{\phi}(x')\hat{\phi}(x)\rangle_0$ satisfies the relationship

$$\tilde{C}_0^>(k) = e^{\beta k_0} \tilde{C}_0^<(k) \qquad (1.53)$$

owing to the KMS condition. Since the KMS condition generally applies, the same relationship can be shown to hold true for interacting 2-point correlation functions.

1.5 Thermal propagators

The most important quantities in thermal field theory are the thermal Green functions

$$G(x_1, x_2, \ldots, x_n) = \langle T\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_n)\rangle , \qquad (1.54)$$

where the symbol T prescribes time ordering. In particular the 2-point thermal Green function for a free system defines the thermal propagator:

$$i\hbar c D(x - x') = \theta(t - t')\langle \hat{\phi}(x)\hat{\phi}(x')\rangle_0 + \theta(t' - t)\langle \hat{\phi}(x')\hat{\phi}(x)\rangle_0 . \qquad (1.55)$$

The function D is called a propagator since it is a Green function of the Klein-Gordon differential operator, that is

$$[(i\partial)^2 - m^2]D(x - x') = \delta(x - x') , \qquad (1.56)$$

as is easily checked by explicit calculation. One of the solutions of this equation is the well known Feynman propagator of zero-temperature field theory

$$D_F(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik\cdot x} . \qquad (1.57)$$

The main difference between zero-temperature and finite temperature field theory lies in the form of the propagators; hence, they require a detailed discussion. As we will show, the fundamental reason is that in thermal equilibrium the propagator has to satisfy the boundary condition prescribed by the KMS condition (1.19).

We may determine a spectral representation for the time-ordered thermal propagator from its definition (1.55) and the previous result (1.48) as

$$iD(x - x') = \int \frac{d^4k}{(2\pi)^4} \rho_0(k) e^{-ik\cdot(x-x')} [\theta(t - t') + n(k_0)] . \qquad (1.58)$$

Like the Feynman propagator, the thermal propagator is a Green function of the inhomogeneous Klein-Gordon equation.

Problem 1.4

Establish that (1.58) is a solution of the inhomogeneous Klein-Gordon equation (1.56).

Putting $\theta(t) = e^{-\epsilon t}$, we calculate the Fourier transform of expression (1.58) as

$$i\tilde{D}(k) = \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi\delta(k^2 - m^2) n(|k_0|) . \qquad (1.59)$$

The remarkable feature of this expression, which was first obtained by Dolan and Jackiw (1973), is that it consists of the sum of the vacuum Feynman propagator and a thermal contribution from the heat bath. Since the particles are free, they are on shell. Note that the absolute value of the energy variable appears in the Bose-Einstein distribution function. This is an awkward feature in calculations.

For reasons that will become clear later on, we also need the thermal propagator for imaginary time $t = -i\tau$, with the imaginary time-coordinate taking values $-\beta\hbar \leq \tau \leq \beta\hbar$. Formally this Euclidean propagator is obtained from the spectral representation (1.58) by setting $t - t' = -i\tau$

$$\begin{aligned} \Delta(\tau, \mathbf{x}) &= iD(-i\tau, \mathbf{x}) \\ &= \int \frac{d^4k}{(2\pi)^4} \rho_0(k) e^{-\tau k_0} e^{i\mathbf{k}\cdot\mathbf{x}} [\theta(\tau) + n(k_0)] . \end{aligned} \qquad (1.60)$$

This propagator is the solution of the inhomogeneous partial differential equation

$$(-\partial_\tau^2 - \nabla^2 + m^2)\Delta(\tau - \tau', \mathbf{x} - \mathbf{x}') = \delta(\tau - \tau')\delta(\mathbf{x} - \mathbf{x}') . \qquad (1.61)$$

By inspection we see that the thermal propagator is periodic in the time variable τ, with period $\beta\hbar$, as a consequence of the KMS condition.

Problem 1.5

Show that $\Delta(\tau - \beta\hbar) = \Delta(\tau)$, if $0 \leq \tau \leq \beta\hbar$, and $\Delta(\tau + \beta\hbar) = \Delta(\tau)$ if $-\beta\hbar \leq \tau \leq 0$.

A useful representation of the thermal propagator is

$$\Delta(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \Delta(\tau, \mathbf{k}) , \qquad (1.62)$$

$$\Delta(\tau, \mathbf{k}) = \frac{1}{2\omega_k} \left\{ [\theta(\tau) + n(\omega_k)] e^{-\tau\omega_k} + [\theta(-\tau) + n(\omega_k)] e^{\tau\omega_k} \right\} , \qquad (1.63)$$

which follows from (1.60) by integrating out the energy variable.

Problem 1.6

Show that the Euclidean thermal propagator in the mixed representation can be written as

$$\Delta(\tau, \mathbf{k}) = \sum_{s=\pm 1} \frac{s}{2\omega_k} [\theta(\tau) + n(s\omega_k)] e^{-s\tau\omega_k} . \qquad (1.64)$$

This representation is very convenient for doing the frequency summations in diagrams.

Since the Euclidean thermal propagator is periodic on the time interval $[0, \beta\hbar]$, we may define a discrete Fourier transform

$$\tilde{\Delta}(i\omega_n, \mathbf{k}) = \int_{a-\beta\hbar}^{a} d\tau\, e^{i\omega_n\tau} \Delta(\tau, \mathbf{k}) , \qquad (1.65)$$

independent of $0 \le a \le \beta\hbar$, defined for discrete so called Matsubara frequencies $\omega_n = 2\pi n(\beta\hbar)^{-1}$, with n integer. Doing the Fourier transform, or more simply, solving equation (1.61) in Fourier space:

$$(\omega_n^2 + \omega_k^2)\tilde{\Delta}(i\omega_n, \mathbf{k}) = 1 , \qquad (1.66)$$

we obtain the Euclidean propagator, also called Matsubara propagator

$$\tilde{\Delta}(i\omega_n, \mathbf{k}) = \frac{1}{\omega_n^2 + \omega_k^2} . \qquad (1.67)$$

We see that the Euclidean thermal propagator looks exactly like a propagator in Minkowski space, with the difference that the variable k_0 takes the discrete and imaginary values $k_0 = i\omega_n$. The Fourier inversion theorem permits the Euclidean propagator (1.60) to be written as the Fourier sum

$$\Delta(\tau, \mathbf{x}) = \frac{1}{\beta\hbar} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_n\tau} \tilde{\Delta}(i\omega_n, \mathbf{k}) . \qquad (1.68)$$

This expression exists for real τ, and is periodic with period $\beta\hbar$.

Problem 1.7

Use (1.63) and (1.68) to show that

$$\frac{1}{\beta\hbar}\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+\omega^2}=\frac{1}{\omega}[1+2n(\omega)] .\qquad(1.69)$$

This is an example of a summation formula for Matsubara frequencies.

It is possible to maintain a useful formal analogy with the vacuum theory by defining a four-vector $k^\mu = (k_0, \mathbf{k})$ with imaginary time component $k_0 = i\omega_n$. In view of (1.68), we may then write the Euclidean propagator as

$$D^E(x) = \frac{1}{\beta\hbar}\sum_n\int\frac{d^3k}{(2\pi)^3}\frac{i}{k^2-m^2}e^{-ik\cdot x} .\qquad(1.70)$$

This expression may be thought of as being obtained from the corresponding vacuum expression by the substitution

$$\int\frac{dk_0}{2\pi i}\to\frac{1}{\beta\hbar}\sum_n\qquad(1.71)$$

with the understanding that the time and the real continuous energy variable must be replaced by $t = -i\tau$, $k_0 = i\omega_n$, respectively.

It is important to stress that the Matsubara propagator (1.67) in Fourier space is not the analytic continuation of the thermal propagator, but of Fourier transforms of the retarded and advanced Green functions.

$$i\hbar D^R(x-x') = \theta(t-t')\langle\left[\hat{\phi}(x),\hat{\phi}(x')\right]\rangle_0 ,\qquad(1.72)$$

$$i\hbar D^A(x-x') = -\theta(t'-t)\langle\left[\hat{\phi}(x),\hat{\phi}(x')\right]\rangle_0 ,\qquad(1.73)$$

which are the causal solutions of the inhomogeneous Klein Gordon equation (1.56). These Green functions play an important role in the description of the dynamical behavior of the system, in particular the propagation of small perturbations. Using the Fourier representation of the spectral function (1.47), we immediately get for the Fourier transform of the retarded and advanced Green functions

$$\tilde{D}^{R,A}(k) = \int\frac{dk_0'}{2\pi}\frac{\rho_0(k_0',\mathbf{k})}{k_0-k_0'\pm i\epsilon} .\qquad(1.74)$$

Integrating over the energy we find the free retarded and advanced Green functions

$$\tilde{D}^{R,A}(k) = \frac{1}{(k_0\pm i\epsilon)^2-\omega_k^2} = \sum_{s=\pm 1}\frac{1}{2\omega_k}\frac{s}{k_0\pm i\epsilon-s\omega_k} .\qquad(1.75)$$

They are temperature independent because the spectral density

$$\rho_0(k) = iD^R(k)-iD^A(k) = \sum_{s=\pm 1}\frac{1}{2\omega_k}2\pi s\,\delta(k_0-s\omega_k)\qquad(1.76)$$

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is essentially the mass shell delta-function. As will be explained later, the physical content of these Green functions is that they give the energies and life times of the collective excitations in the system. In the free case, we simply have free particles and the excitations are of course stable.

To establish the connection of the retarded and advanced propagators with the Matsubara propagator (1.67), it is convenient to write the latter in the spectral form

$$\tilde{\Delta}(i\omega_n, \mathbf{k}) = \int \frac{dk_0}{2\pi} \frac{\rho_0(k)}{k_0 - i\omega_n} \ . \tag{1.77}$$

This expression is only defined at a discrete set of points in the complex energy plane. The analytic extension away from these discrete points such that

$$\tilde{\Delta}(i\omega_n, \mathbf{k}) = \Delta(z, \mathbf{k})|_{z=i\omega_n} \tag{1.78}$$

is not a unique operation without further delimitation. Conventionally one resolves this ambiguity by imposing the requirements:

  (i) $|\Delta(z)| \to 0$ if $|z| \to \infty$

  (ii) $\Delta(z)$ is analytic off the real axis.

These requirements imply that this particular analytic extension has the representation:

$$\Delta(z, \mathbf{k}) = \int \frac{dk_0}{2\pi} \frac{\rho_0(k)}{k_0 - z} \ . \tag{1.79}$$

Comparing with (1.74), we see that $\tilde{D}^{R,A}(k)$ represent the boundary values of the analytic propagator (1.79)

$$\tilde{D}^{R,A}(k) = -\Delta(k_0 \pm i\epsilon, \mathbf{k}) \tag{1.80}$$

as z approaches the real axis from above and below, respectively.

By using the spectral representation (1.74), we can extend the definition of the retarded propagator to any complex energy z such that $\text{Im} z > 0$; it then follows that $D^R(z, \mathbf{k})$ is an analytic function in the upper half-plane, where it coincides with the analytic propagator (1.79), apart from a sign. In the lower half plane, on the other hand, $D^R(z, \mathbf{k})$ is defined by continuation across the real axis, and it may have singularities there. Similarly the advanced propagator can be defined as an analytic function in the lower half plane.

The spectral density is seen to determine both the retarded and advanced propagators, as well as the Matsubara propagator and the thermal propagator. This is a general feature of equilibrium 2-point functions: they all are determined by the spectral density. This also holds true for interacting systems; in fact all spectral representations we have derived so far keep their form for full propagators, with an appropriate spectral density.

The connection between the real-time thermal propagator and the analytic propagator can again be established from the spectral representation. The Fourier transform of (1.58) yields:

$$i\tilde{D}(k) = i \int \frac{dk_0'}{2\pi} \frac{\rho_0(k_0', \mathbf{k})}{k_0 - k_0' + i\epsilon} + \rho_0(k) n(k_0) \ . \tag{1.81}$$

Comparing with (1.79), we see that we may express the time-ordered thermal propagator in terms of the analytic propagator as follows:

$$\tilde{D}(k) = n(k_0)\Delta(k_0 - i\epsilon, \mathbf{k}) - [1 + n(k_0)]\Delta(k_0 + i\epsilon, \mathbf{k}) \tag{1.82}$$

which also relates the thermal propagator to the retarded and advanced Green functions. In this representation the variable k_0 is not restricted to be positive as in (1.59).

1.6 Dirac and gauge fields

The above results may be generalized to fermionic fields with spin. Let us consider the general case of a multi-component covariant complex field $\hat{\psi}^i_\alpha(x)$ which transforms under some representation $\mathcal{D}_{\alpha\beta}$ of the Lorentz group. This field may carry an arbitrary number of charges q^{ij}_a such that

$$\left[\hat{Q}_a, \hat{\psi}^i_\alpha(x)\right] = q^{ij}_a \hat{\psi}^j_\alpha(x) \ , \tag{1.83}$$

with q^{ij}_a an Hermitian matrix in the space of internal degrees of freedom labelled by the indices i, j.

We assume that the free field satisfies a field equation of the form:

$$\Lambda_{\alpha\beta}(i\partial)\hat{\psi}_\beta(x) = 0 \ , \tag{1.84}$$

where $\Lambda_{\alpha\beta}(i\partial)$ is a differential operator of finite order. The equation of motion can only have non-trivial solutions for values of the four-momentum k^μ that satisfy $\det \Lambda(k) = 0$. This dispersion equation determines the energy $k_0 = \pm\omega_k$ of the particles. Since the equation of motion must be consistent with this dispersion relation, there has to exist a non-singular differential operator, called the Klein-Gordon divisor, such that

$$d_{\alpha\beta}(i\partial)\Lambda_{\beta\gamma}(i\partial) = \Lambda_{\alpha\beta}(i\partial)d_{\beta\gamma}(i\partial) = \delta_{\alpha\gamma}(-\partial^2 - m^2) \ . \tag{1.85}$$

In that case the solution of the propagator equation

$$\Lambda_{\alpha\beta}(i\partial)D_{\beta\gamma}(x - x') = \delta_{\alpha\gamma} \ \delta(x - x') \tag{1.86}$$

may be written as

$$D_{\alpha\beta}(x - x') = d_{\alpha\beta}(i\partial)D(x - x') \tag{1.87}$$

in terms of the solution of the Klein-Gordon propagator equation (1.56) satisfying the appropriate boundary conditions.

In the preceding section we have already studied the Klein-Gordon propagator for scalar bosons. It is not difficult to guess that the solution of equation (1.56) in the general case has the form

$$iD(x - x') = \int \frac{d^4k}{(2\pi)^4}\rho_0(k)e^{-ik\cdot(x-x')}[\theta(t - t') + \eta n_+(k_0)] \ . \tag{1.88}$$

The statistical factors $n_\pm = \{\exp[\beta(k_0 \pm \mu)] - \eta\}^{-1}$ are the Fermi-Dirac ($\eta = -1$) or Bose-Einstein ($\eta = 1$) distribution functions for (anti-)particles having a nonzero chemical potential.

For spin -$\frac{1}{2}$ fields, in particular, we have $d(i\partial) = i\gamma \cdot \partial + m$, where the Dirac matrices γ^μ satisfy the algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. The corresponding Euclidean thermal propagator is

$$
\begin{aligned}
S^E(x) &= (i\gamma \cdot \partial + m)D^E(x) & (1.89) \\
&= \frac{i}{\beta\hbar} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{\gamma \cdot k + m}{k^2 - m^2} e^{-ik \cdot x} . & (1.90)
\end{aligned}
$$

It can be shown that for fermionic fields the propagator is anti-periodic in the variable τ with period $\beta\hbar$. As a consequence the Matsubara frequencies are restricted to be proportional to odd integer values: $\omega_n = (2n + 1)\pi(\beta\hbar)^{-1}$. Furthermore, for non-zero chemical potential the energy variable k_0 in (1.90) is shifted by μ, and has the discrete complex values $k_0 = i\omega_n + \mu$. With these rules in mind, all results of the preceding section are easily translated to spin fields.

Gauge fields may be treated in a similar manner and the Euclidean gauge boson and ghost propagators are easily written down without going through any detailed calculations. The gauge propagator for a SU(N) gauge theory with $a = 1, 2, \ldots, N^2 - 1$ color indices, in the Feynman gauge, is particularly simple because all indices "go along for a free ride":

$$
D_{\mu\nu}^{ab}(x - x') = -g_{\mu\nu} \, \delta^{ab} \, D(x - x') . \tag{1.91}
$$

Here $D(x - x')$ is the now familiar Klein-Gordon propagator of the scalar boson field with $m = 0$. Similarly the ghost propagator is

$$
D_{\text{gh}}^{ab}(x - x') = -\delta^{ab} \, D(x - x') . \tag{1.92}
$$

The gauge fields, being bosonic have to satisfy periodic boundary conditions. The ghosts, on the other hand , must be treated as anti-commuting pseudo fermion fields, but subject to the same boundary conditions as the gauge fields. This is a consequence of the fact that the ghosts have to subtract unphysical contributions of the gauge fields. For these boundary conditions the finite temperature action is BRS invariant which ensures that the Slavnov-Taylor identities are easily generalized to finite temperature.

In passing we mention that it would be possible, in principle, to assign anti-periodic boundary conditions to the unphysical components of the gauge field. In that case the ghosts would be a likewise anti-periodic as if they were actual fermion fields. The ghosts would than not represent physical particles in thermal equilibrium, as their only role is to subtract unphysical contributions due to spurious degrees of freedom of the gauge field. However, in practice this procedure is rather awkward and not free of computational difficulties. On the other hand, the procedure is certainly feasible (Landshoff and Rebhan 1992).

Chapter 2

Feynman Path Integral

A path integral is a formal expression for the transition amplitude between two quantum mechanical states in terms of an integration over a certain function space. This provides both a physically intuitive description of the system and a useful starting point for approximations, such as perturbation theory, and loop expansions around stationary points. Since it uses the Lagrangian, rather than the Hamiltonian, as its fundamental entity, the functional formalism explicitly preserves all symmetries of the system. Moreover, the functional approach provides a unified view of quantum field theory and statistical mechanics.

Inspired by the essential idea of Dirac (1933), Feynman (1948) developed the path integral method extensively. The derivation of the Feynman path integral for field theory is based on the concept of coherent states invented by Klauder (1960) which we introduce first.

2.1 Coherent States

For simplicity we shall start with a bosonic Fock space spanned by the base vectors

$$|n\rangle = |n_1, n_2, \ldots\rangle = \prod_k \frac{(\hat{a}_k^\dagger)^{n_k}}{\sqrt{n_k!}} |0\rangle \tag{2.1}$$

obtained by the repeated action of creation operators on the vacuum state $|0\rangle$. For bosons the values of the occupation numbers $n_k = 0, 1, 2, \cdots$, are unrestricted. Adjoint base vectors $\langle n|$ are created by the action of annihilation operators $\hat{a}_k$. These base vectors span the entire Fock space as expressed by the the completeness relation

$$\sum_{\{n\}} |n\rangle\langle n| = 1 . \tag{2.2}$$

Creation and annihilation operators satisfy the commutation relations (1.24). All properties, like the normalization of the base vectors follow from these simple algebraic rules.

Let us now define coherent states. Canonical coherent states are proper eigenstates of $\hat{a}_k$ with continuous complex eigenvalue z_k defined by

$$|z\rangle = N(z)\, e^{z \cdot \hat{a}^\dagger}|0\rangle \ , \qquad (2.3)$$

where $N(z)$ is a normalization factor and the scalar product is shorthand for

$$z \cdot \hat{a}^\dagger = \sum_k z_k \hat{a}_k^\dagger \ . \qquad (2.4)$$

Expanding out and using the definition (2.1) we have

$$|z\rangle = N(z) \sum_n \prod_k \frac{(z_k)^{n_k}}{\sqrt{n_k!}}|n\rangle \ . \qquad (2.5)$$

The eigenvalue is obtained as

$$\hat{a}_k|z\rangle = N(z)\left[\hat{a}_k, e^{z \cdot \hat{a}^\dagger}\right]|0\rangle = z_k|z\rangle \ . \qquad (2.6)$$

The normalization condition

$$
\begin{aligned}
1 = \langle z|z\rangle &= |N(z)|^2 \sum_n \prod_k \frac{(z_k^*)^{n_k}(z_k)^{n_k}}{n_k!} \\
&= |N(z)|^2 e^{z^* \cdot z} \qquad (2.7)
\end{aligned}
$$

determines the normalization factor as

$$N(z) = e^{-\frac{1}{2}z^* \cdot z} \ . \qquad (2.8)$$

The overlap of two coherent states is defined as the inner product

$$
\begin{aligned}
\langle w|z\rangle &= N(w)N(z) \sum_{n,m} \prod_{kl} \frac{w_k^{*n}}{\sqrt{n_k!}}\frac{z_l^{*m}}{\sqrt{m_l!}}\langle n|m\rangle \\
&= \exp(-\tfrac{1}{2}w^* \cdot w - \tfrac{1}{2}z^* \cdot z + w^* \cdot z) \ . \qquad (2.9)
\end{aligned}
$$

The coherent states are over-complete in the sense that two states $|z_1\rangle$ and $|z_2\rangle$ are not linearly independent if $z_1 \neq z_2$. On the other hand, $|z\rangle$ has a finite norm and it is a proper element of the Hilbert space, provided $z^* \cdot z < \infty$.

The important feature of coherent states is that they can be used to resolve the identity. To establish this we need the result

$$\int \frac{dz^* dz}{2\pi i} e^{-z \cdot z^*} z^{*n} z^m = \delta_{nm} n! \ . \qquad (2.10)$$

It then follows that

$$1 = \sum_n |n\rangle\langle n| = \int \prod_k \frac{dz_k^* dz_k}{2\pi i}|z\rangle\langle z| \ . \qquad (2.11)$$

An almost identical computation gives the trace formula

$$\text{Tr}\hat{A} = \int \prod_k \frac{dz_k^* dz_k}{2\pi i} \langle \eta z | \hat{A} | z \rangle . \tag{2.12}$$

We have put in a sign factor $\eta = \pm 1$, because it turns out that for fermions the trace formula is identical except for a minus sign. It is trivial to compute coherent state matrix elements of operators $\hat{A} = A(\hat{a}^\dagger, \hat{a})$ that are in normal order. From the formula (2.5) and its adjoint we immediately obtain:

$$\langle w | : A(\hat{a}^\dagger, \hat{a}) : | z \rangle = A(w^*, z) \langle w | z \rangle , \tag{2.13}$$

where $A(w^*, z)$ is a c-number function.

2.2   Field Theory

In field theory the discrete index k is replaced by the continuous coordinate $\mathbf{x}$ labeling the space points. We consider the real scalar field

$$\hat{\phi}(\mathbf{x}) = \sum_k \left[f_k(\mathbf{x})\hat{a}_k + f_k^*(\mathbf{x})\hat{a}_k^\dagger \right] , \tag{2.14}$$

$$= \hat{\phi}^{(+)}(\mathbf{x}) + \hat{\phi}^{(-)}(\mathbf{x}) , \tag{2.15}$$

which has been decomposed into positive and negative frequency parts. The coherent state is the eigen state of the annihilation part:

$$\hat{\phi}^{(+)}(\mathbf{x})|z\rangle = z(\mathbf{x})|z\rangle , \tag{2.16}$$

with eigenvalue $z(\mathbf{x}) = \sum_k f_k(\mathbf{x})z_k$. The norm generalizes to

$$N(z) = \exp -\tfrac{1}{2} \int d^3x \; [iz^*(\mathbf{x})\dot{z}(\mathbf{x}) - i\dot{z}^*(\mathbf{x})z(\mathbf{x})] , \tag{2.17}$$

where we used (1.27). This shows that this definition only makes sense if $z(\mathbf{x}), \dot{z}(\mathbf{x})$ are in $\text{L}^2(R^3)$, i.e.

$$\int d^3x \; z^*(\mathbf{x})\dot{z}(\mathbf{x}) < \infty . \tag{2.18}$$

Formally the discrete integration measure is replaced by a "continuum product"

$$\prod_k dz_k^* z_k \rightarrow \prod_{\mathbf{x}} dz(\mathbf{x})dz^*(\mathbf{x}) \equiv \mathcal{D}z(\mathbf{x})\mathcal{D}z^*(\mathbf{x}) \tag{2.19}$$

over all space points $\mathbf{x}$. In this way the trace formula (2.12) becomes

$$\text{Tr}A = \int \mathcal{D}z(\mathbf{x})\mathcal{D}z^*(\mathbf{x}) \langle \eta z | \, A \, | z \rangle . \tag{2.20}$$

In general the interpretation of the functional integral is not easy. In this case, however, the integration domain is restricted to functions $z(\mathbf{x})$ in $L^2(R^3)$, which enables us to give a specific definition. Namely, there exits a countable basis $\{\phi_i(\mathbf{x})\}$, so that any function in $L^2(R^3)$ can be written as

$$z(\mathbf{x}) = \sum_i c_i\, \phi_i(\mathbf{x}) \tag{2.21}$$

for certain c_i. Since the ϕ_i's are fixed, the freedom in the choice of $z(\mathbf{x})$ resides in the coefficients c_i, so that the integration measure $\mathcal{D}z(\mathbf{x})\mathcal{D}z^*(\mathbf{x})$ may be replaced by $\prod dc_i dc_i^*$, and we are back at the discrete situation handled before.

2.3   Path Integral

We begin by writing down the transition amplitude from a state $|\varphi_i; t_i\rangle$ at time t_i to the state $|\varphi_f; t_f\rangle$ at time t_f:

$$\langle \varphi_f; t_f | \varphi_i; t_i \rangle = \langle \varphi_f | \, e^{-i\hat{H}(t_f - t_i)} \, | \varphi_i \rangle \;. \tag{2.22}$$

The matrix elements of the exponent cannot be calculated for finite time interval. Thus, the idea is to partition the interval $[t_i, t_f]$ into M infinitesimal pieces $\epsilon = (t_f - t_i)/M$, so that we can write

$$\langle \varphi_f; t_f | \varphi_i; t_i \rangle = \langle \varphi_f | \left(e^{-i\epsilon \hat{H}} \right)^M | \varphi_i \rangle \;. \tag{2.23}$$

We now insert the completeness relation $M-1$ times. Relabeling $|\chi_0\rangle = |\varphi_i\rangle, \langle \chi_M | = \langle \varphi_f |$ we get:

$$\langle \varphi_f; t_f | \varphi_i; t_i \rangle = \int \prod_{n=1}^{M-1} \mathcal{D}\chi_n \mathcal{D}\chi_n^* \prod_{n=1}^{M} \langle \chi_n | e^{-i\epsilon \hat{H}} | \chi_{n-1} \rangle \;. \tag{2.24}$$

The crucial step is to find an approximate expression for the matrix element of the infinitesimal time-evolution operator. One may show that for small ϵ the operator may be approximated by its normal ordered form [NO88]:

$$e^{-i\epsilon \hat{H}} =: e^{-i\epsilon \hat{H}} : + \mathcal{O}(\epsilon^2) \;. \tag{2.25}$$

With this approximation the typical matrix element becomes:

$$\langle \chi_n | \, e^{-i\epsilon \hat{H}} \, | \chi_{n-1} \rangle = \langle \chi_n | \chi_{n-1} \rangle \exp -i\epsilon \, \langle \chi_n | \, \hat{H} \, | \chi_{n-1} \rangle \;. \tag{2.26}$$

The special properties of the coherent states allow us to calculate the matrix element

$$\langle \chi_n | \hat{H}(\hat{a}^\dagger, \hat{a}) | \chi_{n-1} \rangle = H(\chi_n^*, \chi_{n-1}) \;, \tag{2.27}$$

where H is now a function of the c-numbers χ_n and χ_n^*. Collecting the above results we may write for the transition amplitude

$$\langle \varphi_f; t_f | \varphi_i; t_i \rangle = \int \prod_{n=1}^{M-1} \mathcal{D}\chi_n \mathcal{D}\chi_n^* \, e^{iS} \;, \tag{2.28}$$

where the exponent, called the *action*, given by the expression

$$S = \epsilon \sum_{n=1}^{M} \left[\frac{\log\langle\chi_n|\chi_{n-1}\rangle}{i\epsilon} - H(\chi_n^*, \chi_{n-1}) \right] , \qquad (2.29)$$

is a function of the sequence $\{\chi\} = (\chi_1, \chi_2, \cdots, \chi_M)$.

Let us now define a set of intermediate points $t_n = n\epsilon + t_i$ on the time interval $[t_i, t_f]$, so that $t_M = t_f$ and $t_0 = t_i$, and relabel: $\chi_n \to \chi(t_n)$. Then in the limit $M \to \infty$ the sequence of numbers $\{\chi\}$ becomes a function: $\{\chi\} \to \chi(t)$ and the product of integrations a functional measure

$$\sum_{n=1}^{M-1} \mathcal{D}\chi_n \to \mathcal{D}\chi(t) . \qquad (2.30)$$

Unfortunately, the formal expressions are almost meaningless. The reason is that the space of all functions $\chi(t)$ is far too large to be tractable. We cannot assign a meaning to $\mathcal{D}\chi(t)$ like we defined $\mathcal{D}\chi(\mathbf{x})$, because the functions $\chi(\mathbf{x})$ are elements of $L^2(R^3)$, whereas the functions $\chi(t)$ need not be measurable, let alone integrable. The functional integral is only properly defined by the discrete (lattice) form for finite M.

Although we cannot assume continuity or differentiability of $\chi(t)$, we can be bold and restrict the integration domain to a subspace of functions having 'good' mathematical properties. In particular we can demand that this subspace consists of differentiable functions, that is, we assume

$$\chi_n = \chi_{n-1} + \dot{\chi}_n \epsilon + \mathcal{O}(\epsilon^2) \qquad (2.31)$$

for some well defined sequence $\dot{\chi}_n$ which may be called the derivative of the sequence $\{\chi_n\}$. The limit function $\chi(t)$ will then be differentiable:

$$\left.\frac{d\chi}{dt}\right|_{t=t_n} = \lim \frac{\chi_n - \chi_{n-1}}{\epsilon} = \dot{\chi}_n . \qquad (2.32)$$

If this holds we can expand the first term of the action (2.29) as follows:

$$
\begin{aligned}
-i \log\langle\chi_n|\chi_{n-1}\rangle &= \tfrac{1}{2}i \left[\chi_n^* \cdot (\chi_n - \chi_{n-1}) - (\chi_n^* - \chi_{n-1}^*) \cdot \chi_{n-1} \right] \\
&= \tfrac{1}{2}i\epsilon[\chi_n^* \dot{\chi}_n - \dot{\chi}_n^* \chi_n] + \mathcal{O}(\epsilon^2) .
\end{aligned}
\qquad (2.33)
$$

The action (2.29) in the Feynman path integral then takes the form of an action integral

$$\lim_{M \to \infty} S = \int_{t_i}^{t_f} dt\, L(t) \equiv S[\chi^*, \chi] , \qquad (2.34)$$

where $L(t)$ is the Lagrangian

$$L(t) = \tfrac{1}{2}i \left[\chi^*(t) \cdot \partial_t \chi(t) - \partial_t \chi^*(t) \cdot \chi(t) \right] - H\left(\chi^*(t), \chi(t)\right) . \qquad (2.35)$$

Thus, the transition amplitude (2.22) is the sum over all trajectories $\chi(t)$, beginning at $\chi(t_i, \mathbf{x}) = \varphi_i(\mathbf{x})$ and ending at $\chi(t_f, \mathbf{x}) = \varphi_f(\mathbf{x})$,

$$\langle \varphi_f; t_f | \varphi_i; t_i \rangle = \int \mathcal{D}\chi(t) \mathcal{D}\chi^*(t) \, e^{iS[\chi^*, \chi]} \,, \tag{2.36}$$

of the exponential of the action calculated over the finite time interval $[t_i, t_f]$.

In the above derivation we have suppressed the discrete index k of the functions χ_k. Let us now define the following two real c-number functions on configuration space:

$$\phi(\mathbf{x}) = \langle \chi | \hat{\phi}(\mathbf{x}) | \chi \rangle = \sum_k [f_k(\mathbf{x})\chi_k + f_k^*(\mathbf{x})\chi_k^*] \,, \tag{2.37}$$

$$\pi(\mathbf{x}) = \langle \chi | \hat{\pi}(\mathbf{x}) | \chi \rangle = \sum_k [\dot{f}_k(\mathbf{x})\chi_k + \dot{f}_k^*(\mathbf{x})\chi_k^*] \,. \tag{2.38}$$

Using the orthogonality relation (1.27), we may derive from these definitions the identity

$$\tfrac{1}{2} \int d^3x [\pi(\mathbf{x})\delta\phi(\mathbf{x}) - \phi(\mathbf{x})\delta\pi(\mathbf{x})] = \tfrac{1}{2}i \sum_k (\chi_k^*\delta\chi_k - \chi_k\delta\chi_k^*) \tag{2.39}$$

treating χ and χ^* as independent variables. After a partial integration we recognize in the exponent of (2.36) the classical action

$$S = \int_{t_i}^{t_f} dt \int d^3x [\dot{\phi}(x)\pi(x) - \mathcal{H}(x)] \,. \tag{2.40}$$

The final result is the general formula for computing transition amplitudes

$$\langle \varphi_f; t_f | \varphi_i; t_i \rangle = \int \mathcal{D}\phi \mathcal{D}\pi \exp \frac{i}{\hbar} \int_{t_i}^{t_f} dt \int d^3x [\dot{\phi}(x)\pi(x) - \mathcal{H}(x)] \tag{2.41}$$

by means of a path integral. If we specialize to a bosonic scalar field field without derivative coupling, we may split off the Gaussian integral $\mathcal{D}\pi$ over the momentum by completing the square to obtain:

$$\langle \varphi_f; t_f | \varphi_i; t_i \rangle = \int \mathcal{D}\phi \exp \frac{i}{\hbar} \int_{t_i}^{t_f} dt \int d^3x \, \mathcal{L}(x) \tag{2.42}$$

with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\phi(x)(-\partial^2 - m^2)\phi(x) - \mathcal{V}(x) \,. \tag{2.43}$$

The field $\phi(x)$ over which is integrated is constrained to the initial and final configurations $\phi(t_i, \mathbf{x}) = \varphi_i(\mathbf{x})$ and $\phi(t_f, \mathbf{x}) = \varphi_f(\mathbf{x})$, respectively.

Although the expression for the transition amplitude (2.42) has a great formal beauty, we should not forget that it was derived by the rather arbitrary prescription that the sequence χ_n approaches a differentiable function. Even with this prescription, the path integral is mathematically ill defined. The point is that the subspace of smooth functions is so exceedingly small in the total space of functions that it is impossible to find a measure

$\mathcal{D}\phi$ that picks only contributions from these smooth functions. This is the paradox of the continuum path integral: either one restricts it to smooth functions but then it is identically zero or one allows more functions, but then the continuum action is undefined. One way is to go back to the discrete form; this is the method of lattice field theories. Another way is to normalize the formal path integral to an analytically solvable reference problem. In practice this means that the Gaussian path integral for a free system is prescribed.

2.4 Partition function

Let us now consider the partition function for a gas of neutral scalar bosons. We begin by writing the trace in the partition function as a functional integral over a the state space of real functions

$$Z(\beta, V) = \int \mathcal{D}\phi(\mathbf{x}) \langle \phi(\mathbf{x})| \exp\left(-\beta\hat{H}\right) |\phi(\mathbf{x})\rangle . \tag{2.44}$$

The Boltzmann factor may be regarded as the evolution operator that evolves the state from $t = 0$ to the imaginary time $t = -i\hbar\beta$. We may repeat the reasoning of the preceding section, partitioning the interval $[0, \hbar\beta]$ into M infinitesimal pieces $\epsilon = \hbar\beta/M$. We define a set of intermediate points $\tau_n = n\epsilon$ on the imaginary time interval $[0, \hbar\beta]$, so that $\tau_M = \beta$ and $\tau_0 = 0$. Then in the limit $M \to \infty$, the partition function becomes a path intergral

$$Z = \int \mathcal{D}\phi(\tau, \mathbf{x}) \exp -\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau L^E[\phi] . \tag{2.45}$$

over the field $\phi(\tau, \mathbf{x})$ defined on the Euclidean time inteval $0 \le \tau \le \hbar\beta$. The fact that in the trace the initial and final states are the same, requires the Euclidean field to satisfy the periodic boundary condition:

$$\phi(0, \mathbf{x}) = \phi(\hbar\beta, \mathbf{x}) \tag{2.46}$$

at both ends of the imaginary-time interval. The Euclidean Lagrangian is related to the classical Lagrangian density (2.86) in Minkowski space through

$$L^E(\tau) = -\int d^3x \, \mathcal{L}(-i\tau, \mathbf{x}) , \tag{2.47}$$

with the time t analytically continued to $-i\tau$.

In summary, the partition function (2.44) can be represented as an Euclidean functional integral over fields $\phi(\tau, \mathbf{x})$ defined on the time interval $t = -i\tau, 0 \le \tau \le \hbar\beta$. The path integration is subject to the periodicity condition (2.46).

The path integral (2.45) is defined up to an unspecified and ill-defined normalization factor. However, for the calculation of thermodynamic functions this normalization factor is irrelevant as we will now show for the example of the energy density

$$\begin{aligned} E &= -\frac{1}{V} \frac{\partial \log Z}{\partial \beta} \\ &= \frac{1}{V} \left\langle \frac{\partial}{\partial \hbar\beta} \int_0^{\hbar\beta} d\tau L^E[\phi] \right\rangle . \end{aligned} \tag{2.48}$$

As we shall show, the term written out is indeed the energy density, implying that terms coming from the β-dependence of the normalization constant and the integration measure $\mathcal{D}\phi(\tau, \mathbf{x})$ must cancel. To that purpose we write $\tau = \hbar\beta\lambda$, $0 \leq \lambda \leq 1$. After replacing τ by λ we easily calculate the β-derivative in (2.48). The result is

$$E = \frac{1}{\hbar\beta V} \int_0^{\hbar\beta} d\tau \int d^3x \left\langle (\partial_\tau \phi)^2 - \mathcal{L} \right\rangle . \tag{2.49}$$

Since the equilibrium average does not depend on the space-time coordinates, we get

$$E = \langle \mathcal{H}(x) \rangle , \tag{2.50}$$

where $\mathcal{H}$ is the Hamiltonian density of the system. We conclude that (2.45) is indeed a valid representation for the canonical partition function.

We now consider partition function for the non-interacting case. Then, we have to calculate the Gaussian path intgral

$$Z_0 = \int \mathcal{D}\phi \exp -\tfrac{1}{2}\phi\Lambda\phi , \tag{2.51}$$

where the exponent reads explicitly

$$\tfrac{1}{2}\phi\Lambda\phi = \frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \int d^3x \, \phi(\tau, \mathbf{x}) \left(-\partial_\tau^2 - \nabla^2 + m^2 \right) \phi(\tau, \mathbf{x}) . \tag{2.52}$$

For bosons the general formula for the Gaussian integral is

$$\int \mathcal{D}\phi \exp -\tfrac{1}{2}\phi\Lambda\phi = (\det \Lambda)^{-\frac{1}{2}} . \tag{2.53}$$

We may use the identity $\det \Lambda = \exp \operatorname{Tr} \log \Lambda$, which is obvious for any matrix that can be diagonalized. For the partition function this gives

$$\log Z_0 = -\tfrac{1}{2}\operatorname{Tr} \log \Lambda , \tag{2.54}$$

where Λ is to be regarded as a matrix in an infinite dimensional function space labeled by $\tau, \mathbf{x}$.

To get further it is convenient to introduce an expansion in plane-wave eigenfunctions of the operator Λ

$$\delta(\tau - \tau')\delta(\mathbf{x} - \mathbf{x}') = \frac{1}{\hbar\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \exp -i\omega_n(\tau - \tau') \exp i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') , \tag{2.55}$$

where the Matsubara frequencies $\omega_n = 2\pi nT/\hbar$ are determined by the periodic boundary conditions on the boson field. The logarithm is defined by its series expansion in powers of Λ. The first term is calculated as

$$
\begin{aligned}
\operatorname{Tr}\Lambda &= \frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \int d^3x \, \Lambda \, \delta(\tau - \tau')\delta(\mathbf{x} - \mathbf{x}')|_{\tau=\tau', \mathbf{X}=\mathbf{X}'} \\
&= \beta V \operatorname{Tr}(\omega_n^2 + \omega_k^2) ,
\end{aligned}
\tag{2.56}
$$

where the argument is the inverse of the propagator (3.31) for bosons at temperature T. The trace in the last line of (2.56) must be understood as

$$\mathrm{Tr} \equiv \frac{1}{\hbar\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \ . \tag{2.57}$$

Similarly we get for the quadratic term

$$\mathrm{Tr}\Lambda^2 = \beta V \, \mathrm{Tr}(\omega_n^2 + \omega_k^2)^2 \tag{2.58}$$

and so forth. As it should be, the logarithm of the partition function is an extensive quantity proportional to the volume. The thermodynamic potential is therefore given by

$$\Omega = \tfrac{1}{2}\beta \, \mathrm{Tr} \log(\omega_n^2 + \omega_k^2) \ . \tag{2.59}$$

One may note that the sum over n is divergent; moreover the dimension of the right-hand side is not correct. This is the result of not properly taking into account the normalization factor. Nevertheless, the final answer will come out all right.

The summation over the Matsubara frequencies can be performed by contour integration. However, it is simpler to differentiate with respect to the mass

$$\frac{1}{2m}\frac{\partial\Omega}{\partial m} = \tfrac{1}{2}\beta \, \mathrm{Tr} \frac{1}{\omega_n^2 + \omega_k^2} \tag{2.60}$$

because we can then use the summation formula (1.69).

Problem 2.1

> *Integrate the above formula with respect to m and rederive the expression for the thermodynamic potential (1.45) for an ideal gas of bosons.*

2.5   Dirac Field

A similar derivation of the path integral can be given for fermionic fields. Because fermionic fields anti-commute, it turns out to be necessary to replace the complex numbers z, z^* by new quantities, called Grassman variables, that satisfy different calculational rules. In particular they anti-commute among themselves, and commute with all bosonic quantities, like bosonic operators and ordinary c-numbers. The point of these rules is that the whole coherent state formalism, including the derivation of the path integral, can be taken over almost literally. The only essential difference is that, on account of the minus sign in the trace formula (2.20), the boundary condition on fermionic fields in the path integral is anti-periodic: $\psi(\hbar\beta, \mathbf{x}) = -\psi(0, \mathbf{x})$.

We shall now write down the path integral representation for the partition function

$$Z = \mathrm{Tr} e^{-\beta(\hat{H} - \mu\hat{Q})} \tag{2.61}$$

for a system of free Dirac particles with Hamiltonian

$$\hat{H} = \int d^3x \hat{\psi}^\dagger(x)\gamma^0(-i\gamma\cdot\nabla + m)\hat{\psi}(x) \tag{2.62}$$

and charge

$$\hat{Q} = \int d^3x\, \hat{\psi}^\dagger(x)\gamma^0\hat{\psi}(x) \ . \tag{2.63}$$

The charge is conserved and satisfies the commutation relations:

$$\left[\hat{Q}, \hat{\psi}(x)\right] = -\hat{\psi}(x), \quad \left[\hat{Q}, \hat{\psi}^\dagger(x)\gamma^0\right] = \hat{\psi}^\dagger(x)\gamma^0 \ . \tag{2.64}$$

We replace the operator fields $\hat{\psi}_\alpha(\mathbf{x})$ by Grassman variables $\psi_\alpha(\tau, \mathbf{x})$ defined on the Euclidean time interval, $t = -i\tau, 0 \le \tau \le \beta$, and construct the corresponding Euclidean action

$$Z = \int \mathcal{D}\psi^*\mathcal{D}\psi \exp - \int_0^\beta d\tau \left[\int d^3x \psi_\alpha^*(\partial_\tau - \mu)\psi_\alpha + H(\psi^*, \psi)\right] \ . \tag{2.65}$$

The functional integration over the Grassman variables is subject to the anti-periodicity conditions:

$$\psi_\alpha^*(0, \mathbf{x}) = -\psi_\alpha^*(\beta, \mathbf{x}), \quad \psi_\alpha(0, \mathbf{x}) = -\psi_\alpha(\beta, \mathbf{x}) \tag{2.66}$$

We deduce that the thermal propagator is given by

$$S(\tau - \tau', \mathbf{x} - \mathbf{x}') = \langle\psi(\tau, \mathbf{x})\bar{\psi}(\tau', \mathbf{x}')\rangle_0 \tag{2.67}$$

and obeys the partial differential equation

$$[\gamma^0(\partial_\tau - \mu) - i\gamma\cdot\nabla + m]S(\tau - \tau', \mathbf{x} - \mathbf{x}') = \delta(\tau - \tau')\delta(\mathbf{x} - \mathbf{x}') \ . \tag{2.68}$$

Problem 2.2

Show that the solution of this equation in momentum space is the Matsubara popagator for fermions

$$\tilde{S}(k) = (\gamma\cdot k + m)\tilde{D}(k) \ , \tag{2.69}$$

where the energy has the discrete values $k_0 = i\omega_n - \mu, \omega_n = \pi(2n + 1)T$.

The basic Gaussian integration formula for a Grassman field is

$$\int \mathcal{D}\psi^*\mathcal{D}\psi \exp -\psi^*\Lambda\psi) = \det\Lambda \ . \tag{2.70}$$

With $\det\Lambda = \exp\mathrm{Tr}\log\Lambda$, we get for the partition function of the free fermion gas

$$\log Z_0 = \mathrm{Tr}\Lambda = \mathrm{Tr}\log[(\partial_\tau - \mu) - i\gamma^0\gamma\cdot\nabla + \gamma^0 m] \ . \tag{2.71}$$

The trace operation is to be carried out both over the Dirac indices and in momentum -frequency space.

Problem 2.3

Calculate the trace to obtain

$$\Omega = -\sum_n \int \frac{d^3k}{(2\pi)^3} \left\{ \log[\omega_n^2 + (\omega_k - \mu)^2] + \log[\omega_n^2 + (\omega_k + \mu)^2] \right\} \tag{2.72}$$

for the thermodynamic potential.

We may now use a summation formula analogous to (1.69), which can easily be derived from the basic thermal propagator (1.88), to obtain the thermodynamic potential of a Fermi gas

$$\Omega = -2 \int \frac{d^3k}{(2\pi)^3} \left\{ \log[1 + e^{-\beta(\omega_k - \mu)}] + \log[1 + e^{-\beta(\omega_k + \mu)}] + \beta\omega_k \right\} . \tag{2.73}$$

It has an overall factor two owing to the two spin states.

2.6   Generating functional

We consider again a scalar Bose field $\hat{\phi}(x)$ carrying no conserved charges. Thermal Green functions are defined as expectation values of time-ordered fields in the Heisenberg picture:

$$\begin{aligned} G(x_1, \ldots, x_N) &= \langle T\hat{\phi}(x_1) \ldots \hat{\phi}(x_N) \rangle \\ &= \frac{1}{Z} \int \mathcal{D}\varphi(\mathbf{x}) \langle \varphi(\mathbf{x}); t_i | e^{-\beta\hat{H}} T\hat{\phi}(x_1) \ldots \hat{\phi}(x_N) | \varphi(\mathbf{x}); t_i \rangle . \end{aligned} \tag{2.74}$$

The trace is over a complete set of coherent states at some arbitrary initial time t_i. By a spectral analysis it may be shown that thermal Green functions exist as an analytic function of their time-variables $t_1, \ldots, t_N$ as long as the imaginary parts of the time differences satisfy

$$-\beta < \mathrm{Im}(t_i - t_j) < 0 \quad , \tag{2.75}$$

with t_i the larger and t_j the smaller time. Basically the argument is that the Boltzmann factor in the thermal average ensures convergence of the trace.

It is convenient to collect all Green functions in the generating functional

$$Z[j] = \int \mathcal{D}\varphi(\mathbf{x}) \langle \varphi(\mathbf{x}); t_i | e^{-\beta\hat{H}} \, T \exp i \int d^4x \, j(x)\hat{\phi}(x) \, | \varphi(\mathbf{x}); t_i \rangle \quad , \tag{2.76}$$

where $j(x)$ is a *c*-number source. The individual Green functions are recovered by expanding out the generating functional. The first term is the canonical partition function

$$Z[0] = \int \mathcal{D}\varphi(\mathbf{x}) \langle \varphi(\mathbf{x}); t_i | e^{-\beta\hat{H}} | \varphi(\mathbf{x}); t_i \rangle \quad . \tag{2.77}$$

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Higher-order Green functions are obtained by repeated functional differentiation with respect to the source:

$$\langle\, T\hat{\phi}(x_1)\hat{\phi}(x_2)\ldots\hat{\phi}(x_N)\,\rangle = \frac{1}{Z[0]}\frac{\delta^N Z[j]}{i\delta j(x_1)\ldots i\delta j(x_N)}\bigg|_{j=0}\,. \qquad (2.78)$$

When calculating these N-point thermal Green functions, one finds that in general they consist of various connected pieces, that is, sub-units that have the cluster property, namely

$$\lim_{|x_i-x_j|\to\infty} G_{\mathrm{con}}(x_1, x_2, \ldots, x_N) = 0 \qquad (2.79)$$

for any two arguments x_i, x_j. That is, the connected pieces fall off to zero when the distance between the field points increases. Connected N-point Green functions, for which we use the notation $G^{(N)}$ in the following, are obtained from the logarithm of the generating functional

$$G^{(N)}(x_1, x_2, \ldots, x_n) = \frac{\delta^N \log Z[j]}{i\delta j(x_1)\ldots i\delta j(x_N)}\bigg|_{j=0}\,. \qquad (2.80)$$

The expansion of the new generating functional $W[j] = \log Z[j]$ in terms of connected Green functions

$$W[j] = \log Z[0] + \sum_{N=1}^{\infty}\frac{i^N}{N!}\int d^4x_1\ldots d^4x_N\, G^{(N)}(x_1, \ldots, x_N)j(x_1)\ldots j(x_N)\,. \qquad (2.81)$$

is usually called the cumulant expansion in statistical mechanics. The first few connected Green functions read:

$$\begin{aligned} G^{(1)}(x) &= \langle\hat{\phi}(x)\rangle & (2.82)\\ G^{(2)}(x_1, x_2) &= \langle\, T\hat{\phi}(x_1)\hat{\phi}(x_2)\,\rangle - \langle\,\hat{\phi}(x_1)\,\rangle\langle\,\hat{\phi}(x_2)\,\rangle\,. & (2.83) \end{aligned}$$

Higher-order expressions may be found in [**?**]. It is preferable to work with connected Green functions; not only are they diagrammatically simpler objects, but also they have the cluster property (2.79).

Problem 2.4

Show that for translationally invariant systems, thermal Green functions $G^{(N)}(x_1, \ldots, x_N)$ effectively only depend on $N-1$ relative coordinates.

We now recall the general formula (2.42) that expresses transition amplitudes in terms of a path integral. The derivation is easily generalized to matrix elements of a product of time-ordered fields with time-arguments in the time-interval $[t_i, t_f]$. This generalization leads to the Feynman-Matthews-Salam (FMS) formula

$$\langle\varphi_f; t_f|\, T\hat{\phi}(x_1)\ldots\hat{\phi}(x_N)\,|\varphi_i; t_i\rangle = \int\mathcal{D}\phi\,\phi(x_1)\ldots\phi(x_N)\,e^{iS} \qquad (2.84)$$

with action

$$S = \int_{t_i}^{t_f} dt \int d^3x \, \mathcal{L}(x) \tag{2.85}$$

in terms of the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\phi(x)(-\partial^2 - m^2)\phi(x) - \mathcal{V}(x) \ . \tag{2.86}$$

The path integration is over all c-number fields $\phi(x)$ constrained to the initial and final configurations $\phi(t_i, \mathbf{x}) = \varphi_i(\mathbf{x})$ and $\phi(t_f, \mathbf{x}) = \varphi_f(\mathbf{x})$, respectively. The time-arguments $t_1, \ldots, t_N$ must all lie in the interval $[t_i, t_f]$.

For the application of the FMS-formula to thermal field theory, it is crucial to understand that the result (2.84) retains its validity if we allow the times to be complex. More precisely, let C be some oriented contour $t = z(\tau)$ in the complex time plane beginning at t_i and ending at t_f, with the parameter τ real and monotonically increasing; and let T_C be an ordering instruction that rearranges operators in the Heisenberg picture in the order in which their arguments lie along the oriented contour C, the later times to the left. (Note that on the real-time axis T_C-ordering amounts to ordinary time ordering with respect to τ.) The only restriction is that the contour C must go monotonically downward, as expressed by the inequality (2.75). Since the limit of an analytic function on the boundary of its domain of definition, where it is still continuous, exists as a generalized function, in the limit case the contour may be parallel the real axis. Hence, the imaginary part of a point moving along the contour C must be non-increasing. Of course, for the FMS-formula (2.84) to be valid, the time arguments of the fields must all lie on the contour C. To see that this reasoning is viable it suffices to point out that any contour allows a decomposition into pieces, each parameterized by a real parameter, such that contour ordering coincides with ordinary time-ordering in this real parameter. The original FMS-formula can than be applied piecewise.

Let us now return to the generating functional (2.76). The action of the canonical operator may be translated into an imaginary time shift

$$\langle \varphi(\mathbf{x}); t_i | \, e^{-\beta \hat{H}} = \langle \varphi(\mathbf{x}); t_i - i\beta | \ . \tag{2.87}$$

Hence, to be able to apply the FMS-formula, we have to consider a contour C in the complex time plane which starts at t_i and ends at $t_f = t_i - i\beta$. If the time-arguments of the fields lie on the real axis, the contour has to include these as well. The FMS-formula allows us then to write the generating functional as the path integral

$$Z[j] = \int \mathcal{D}\phi \exp i \int_C d^4x [\mathcal{L}(x) + j(x)\phi(x)] \ , \tag{2.88}$$

where the path integration is over all c-number fields $\phi(x)$ which have to satisfy the periodic boundary condition

$$\phi(t_i - i\beta, \mathbf{x}) = \phi(t_i, \mathbf{x}) \ , \tag{2.89}$$

because the trace in the thermal average demands that the fields at the begin and end points are the same.

Chapter 3

Imaginary-Time Formalism

A major part of the applications of Thermal Field Theory relies on some version of thermal perturbation theory or other. Still the most popular is the imaginary-time (Euclidean) formalism (ITFT). The formalism was developed by many authors , but is usually named after Matsubara (1953), who was the first top set up a diagrammatic perturbation theory for the grand partition function on a field theoretic basis. The major advantage of ITFT is that in Fourier language the Feynman rules are very similar to those of the vacuum theory, except that the energies in the propagators are discrete and imaginary. The Green functions of the theory are defined at these Matsubara frequencies, which constitute a discrete set of points on the imaginary axis of the complex energy plane. The formalism is well suited to the evaluation of static thermodynamic properties, e.g. the thermodynamic potential. Dynamical problems, on the other hand, necessitate an analytic extension of the Matsubara Green functions to the real axis, owing to the unphysical representation of energy and time. This is the basic disadvantage of the Matsubara formalism.

3.1 Wick's Theorem

As explained in the preceding chapter, in the path integral of thermal field theory the time integrations have to performed along a contour from some arbitrary time t_i down to $t_i - i\beta$. In this chapter we confine ourselves to a time-contour that runs straight down the imaginary time axis $t = -i\tau, 0 \leq \tau \leq \beta$, with all time arguments of the fields on this imaginary time contour; we write $\phi(\tau, \mathbf{x}) \equiv \phi(-i\tau, \mathbf{x})$. On this Euclidean contour the action in the path integral (5.6) becomes the Euclidean action

$$i \int_C d^4x \, \mathcal{L}(x) = - \int_0^\beta d\tau \int d^3x \, \mathcal{L}^E(x) \equiv -S^E \ . \tag{3.1}$$

From (2.86) we obtain the Euclidean Lagrangian density

$$\mathcal{L}^E(\tau, \mathbf{x}) = \tfrac{1}{2}\phi(\tau, \mathbf{x}) \left(-\partial_\tau^2 - \nabla^2 + m^2 \right) \phi(\tau, \mathbf{x}) + \mathcal{V}(\phi) \ . \tag{3.2}$$

The interaction term is arbitrary, except that it can not contain derivative interactions, i.e., it can not depend on derivatives of ϕ. With these specifications the generating functional (5.6) on the Euclidean contour takes the form

$$Z[j] = \int \mathcal{D}\phi \exp(-S^E + j\phi) , \qquad (3.3)$$

where we used the shorthand notation

$$j\phi = \int_0^\beta d\tau \int d^3x \; j(\tau, \mathbf{x})\phi(\tau, \mathbf{x}) . \qquad (3.4)$$

One may note this is very similar to the generating functional of a $T = 0$ field theory in Euclidean space, except that the integration over the Euclidean time variable τ would then run over the range $(-\infty, \infty)$.

We proceed by considering the generating functional for the free case with $\mathcal{V} = 0$. With the same notation as in formula (3.8) we write

$$Z_0[j] = \int \mathcal{D}\phi \exp -\tfrac{1}{2}\phi\Lambda\phi + j\phi , \qquad (3.5)$$

where $\Lambda = -\partial_\tau^2 - \nabla^2 + m^2$ is the differential operator determined by the quadratic part of the Lagrangian. The integration over the field is Gaussian and can be performed by transcribing formula (6.106) to the continuum case

$$Z_0[j] = Z_0[0] \; \exp \tfrac{1}{2}j\Lambda^{-1}j . \qquad (3.6)$$

The inverse of the differential operator Λ is the propagator defined through the Green function equation

$$(-\partial_\tau^2 - \nabla^2 + m^2)\Delta(\tau - \tau', \mathbf{x} - \mathbf{x}') = \delta(\tau - \tau')\delta(\mathbf{x} - \mathbf{x}') . \qquad (3.7)$$

A unique solution is obtained by prescribing the appropriate boundary condition which in this case is that the propagator must be periodic on the time interval $0 \le \tau \le \beta$. This implies that the solution is the thermal propagator for imaginary time (1.60) as defined in chapter 1. Hence, for the generating functional we get:

$$Z_0[j] = Z_0[0] \; \exp \tfrac{1}{2}\int_0^\beta d\tau d\tau' \int d^3x d^3x' \; j(\tau, \mathbf{x})\Delta(\tau - \tau', \mathbf{x} - \mathbf{x}')j(\tau', \mathbf{x}') , \qquad (3.8)$$

with the free partition function as calculated in chapter 2. The correspondence with the vacuum theory is easily memorized through the transcription rules

$$i\int dt \;\; \rightarrow \;\; \int_0^\beta d\tau \qquad (3.9)$$

$$iD_F(x) \;\; \rightarrow \;\; \Delta(\tau, \mathbf{x}) \qquad (3.10)$$

with the Feynman propagator $D_F(x)$ as defined in (1.57).

The implication of (3.8) is that the free generating functional may be expanded entirely into products of the thermal propagator $\Delta(\tau, \mathbf{x})$. This is called the thermodynamic Wick theorem. In particular, differentiating twice with respect to the source and setting $j = 0$ afterwards we obtain the basic expectation value of two free fields with time arguments on the imaginary time contour:

$$
\begin{aligned}
\langle \phi(\tau, \mathbf{x})\phi(0) \rangle_0 &= \Delta(\tau, \mathbf{x}) \\
&= \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3 k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_n \tau}}{\omega_n^2 + \omega_k^2} \; .
\end{aligned}
\tag{3.11}
$$

This may be generalized to any thermal average of an even number of fields which may be reduced to a sum of products of propagators

$$
\langle \phi(x_1) \ldots \phi(x_l) \rangle_0 = \sum_{\substack{\text{pairings of} \\ (k_1, \ldots, k_l)}} \Delta(x_{k_1} - x_{k_2}) \ldots \Delta(x_{k_{l-1}} - x_{k_l}) \; ,
\tag{3.12}
$$

where $x = (\tau, \mathbf{x})$. The sum runs over all possible ways of choosing coordinate pairs. In field theory this procedure is often referred to as "contracting" the coordinate pairs. Each contraction is represented by a propagator at the right hand side of (3.12).

3.2   Perturbation expansion

The path-integral representation for the generating functional generates perturbative expansions by the following algorithm. First we split the Euclidean action into a quadratic part and a remainder: $S^E = S_0 + S_I$. For the example of a $\lambda \phi^4$-theory these two pieces have the form:

$$
\begin{aligned}
S_0[\phi] &= \tfrac{1}{2} \int_0^\beta d\tau \int d^3 x \; \phi(x)(-\partial_\tau^2 - \nabla^2 + m^2)\phi(x) \; ,
\tag{3.13} \\
S_I[\phi] &= \frac{\lambda}{4!} \int_0^\beta d\tau \int d^3 x \; \phi(x)^4 \; ,
\tag{3.14}
\end{aligned}
$$

where λ is the coupling constant. Since the action as defined here is dimensionless, $[S^E] = 0$, the field, mass and coupling constant have the dimensions: $[\phi] = 1, [m] = 2, [\lambda] = 0$, with scale -1 attributed to the length scale L.

Wick's theorem allows the systematic calculation of any thermal Green function derivable from the generating functional (6.15) which we write as

$$
Z[j] = \int \mathcal{D}\phi \; e^{-S_0[\phi]} e^{-S_I[\phi] + j\phi} \; .
\tag{3.15}
$$

This expression may be regarded as a statistical average of the second exponential factor with statistical weight $\exp -S_0[\phi]$. By expanding out the second exponent one gets

$$
Z[j] = Z_0[0]\{1 + \langle -S_I + j\phi \rangle_0 + \tfrac{1}{2}\langle (-S_I + j\phi)^2 \rangle_0 + \ldots\} \; .
\tag{3.16}
$$

Any two fields are contracted and replaced by the propagator (3.11). It proves to be extremely convenient to introduce a graphical notation:

(i) a contraction is represented by a line joining the two arguments

$$\langle \phi(x)\phi(y)\rangle_0 = \Delta(x-y) = \underline{\hspace{3cm}} \tag{3.17}$$

(ii) the four-interaction is represented by a four-point vertex
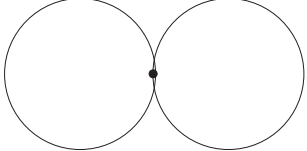
$$-\lambda\phi(z)^4 = \quad \times \quad \tag{3.18}$$

(iii) an external source attached to a field $\phi(x)$ is marked by a cross:

$$j(x) = ----\times \tag{3.19}$$

(iv) all vertex- and source coordinates are integrated over: $\int d\tau \int d^3x$

In this manner any term in the Wick expansion of (3.16) can be represented by a unique *Feynman diagram*. For example

$$\langle S_I\rangle_0 \; = \; \frac{\lambda}{4!}\int_0^\beta d\tau \int d^3x \langle \phi(x)^4\rangle_0 = \frac{\lambda}{8}\int_0^\beta d\tau \int d^3x \left[\Delta(0)\right]^2$$

$$= \quad \infty \quad . \tag{3.20}$$

Such a diagram without external point is called a closed or bubble diagram. Closed graphs are not connected to a source and do not contribute to correlation functions. The sum of all closed diagrams equals the partition function:

$$Z[0] = Z_0[0]\sum_{n=0}^\infty \frac{(-\lambda)^n}{n!}\langle (S_I)^n\rangle_0 \;, \tag{3.21}$$

where we have extracted the coupling constant from the action to exhibit the order of the terms in the perturbation series. Hence, what actually has to be computed at order λ^n is

$$\langle (S_I)^n\rangle_0 = \frac{\int \mathcal{D}\phi e^{-S_0}(S_I)^n}{\int \mathcal{D}\phi e^{-S_0}} \;, \tag{3.22}$$

which is the value of S_I raised to an arbitrary power and averaged over the unperturbed ensemble represented by S_0. The ill-defined normalization of the integrals is irrelevant because we only need their ratio.

In the preceding chapter we already introduced the notion of a cumulant expansion. In the present case cumulants may be formally defined by writing:

$$Z[0] = Z_0[0] \exp \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \langle (S_I)^n \rangle_{\text{con}} .$$ (3.23)

The cumulants may be identified by expanding out both (3.21) and (3.23) with respect to the coupling constant, and identifying terms of equal power. By working out a few examples it becomes clear that the cumulant construction is such that all disconnected pieces from any term $\langle (S_I)^n \rangle_0$ are subtracted, leaving only terms that are represented by connected diagrams. Hence we may write for the logarithm of the partition function (3.21), which is the quantity of physical interest in statistical mechanics:

$$\log Z[0] = \log Z_0[0] + \langle e^{-\lambda S_I} - 1 \rangle_{\text{con}} ,$$ (3.24)

where the last term stands for the sum of all connected closed diagrams.

The cumulants represented by closed linked diagrams have two very important characteristic properties, namely:

- they have the cluster property, and

- they are extensive, that is, they are proportional to the volume of the system.

The last property follows immediately from Wick's theorem (3.12). Since the propagators only depend on the coordinate differences, in terms represented by a closed linked diagram, one of the coordinates is redundant and the final integration will yield a factor βV. Every disconnected closed piece of a diagram, that is, every piece that is not connected to an external point will yield such a factor.

Recalling now the basic statistical mechanical formula for the pressure $P = \log Z / \beta V$, we see that we have derived a general diagram technique for evaluating the interaction part $P_I = P - P_0$ of the pressure

$$P_I = \frac{1}{\beta V} \log \frac{Z[0]}{Z_0[0]} = \frac{1}{\beta V} \sum \text{closed linked diagrams} .$$ (3.25)

This expression remains finite in the thermodynamic limit, as a consequence of the translational invariance of the underlying field theory.

Problem 3.1

Draw the first- and second-order diagrams contributing to the thermodynamic pressure.

We carry on with the full generating functional. Since the coupling of the external source to the field is just another type of vertex, we may follow the same reasoning and write

$$\begin{aligned} \log Z[j] &= \log Z_0[0] + \langle e^{-\lambda S_I + j\phi} - 1 \rangle_{\text{con}} , \\ &= \log Z[0] + \langle e^{-\lambda S_I} (e^{j\phi} - 1) \rangle_{\text{con}} , \end{aligned}$$ (3.26)

where the last term at the right-hand side is the sum of all connected diagrams with one or more external points. For example, for the 2-point thermal Green function we have the expression:

$$G^{(2)}(x, y) = \langle e^{-\lambda S_I} \phi(x)\phi(y) \rangle_{\text{con}} .$$
(3.27)

Working out the terms with Wick's theorem, we find each term in the perturbative expansion as a product of propagators $\Delta(x_i - x_j)$, represented by a connected Feynman diagram.

Problem 3.2

Consider (3.27) and write down the perturbative expansion to order λ with the correct numerical symmetry factor.

The only factor that is not completely obvious is the multiplicative numerical factor, called the *symmetry factor* that we have to assign to each diagram. The symmetry factor is obtained in a straightforward, but tedious, manner by counting the number of ways in which a diagram can be constructed by connecting the vertices with the same topological result. Let the diagram consist of V vertices and the corresponding symmetry factor be S_V. In the perturbation expansion (3.20) it is seen that this symmetry factor has to be divided by the permutational factor of each vertex: $(4!)^V$, and by the permutation of the identical vertices: $V!$. In total we get the weight factor:

$$g_V = \frac{S_V}{(4!)^V V!} .$$
(3.28)

Since the details of counting diagrams is treated in many textbooks, e.g. [PS95], we will not pursue this here.

3.3 Feynman rules

In the imaginary-time formalism, the Feynman rules take their simplest form in momentum space. In virtue of the periodicity conditions, the time dependence of all fields may be represented by a Fourier sum over discrete frequencies. In general we write:

$$f(x) = \frac{1}{\beta} \sum_n \int \frac{d^3 k}{(2\pi)^3} \exp(-ik \cdot x) \tilde{f}(k)$$
(3.29)

where $x_0 = -i\tau$ and k_0 takes on the discrete values $k_0 = i2n\pi\beta^{-1} + \mu$ or $k_0 = i(2n + 1)\pi\beta^{-1} + \mu$, depending on the periodicity properties of the field $f(x)$. In terms of the Fourier transformed fields the action becomes:

$$i \int_0^{-i\beta} d^4 x \mathcal{L}(x) = -\frac{1}{2\beta} \sum_n \int \frac{d^3 k}{(2\pi)^3} \tilde{\phi}(-k) \tilde{\Delta}^{-1}(k) \tilde{\phi}(k) + S_I[\tilde{\phi}] .$$
(3.30)

Here

$$\tilde{\Delta}(k) = \frac{1}{m^2 - k^2} = \frac{1}{\omega_n^2 + \omega_k^2} \tag{3.31}$$

is the thermal (Matsubara) propagator and

$$S_I[\tilde{\phi}] = -\frac{\lambda}{4!} \prod_{i=1}^{4} \frac{1}{\beta} \sum_{n_i} \int \frac{d^3 k_i}{(2\pi)^3} \tilde{\phi}(k_i) \beta (2\pi)^3 \delta_{n,0} \delta(\mathbf{k}) , \tag{3.32}$$

with $\mathbf{k} = \sum_i \mathbf{k}_i$ and $n = \sum_i n_i$. On account of the locality of the interaction, energy and momentum are conserved at each vertex.

The Fourier transformed Green functions

$$\tilde{G}^{(N)}(k_1, \dots, k_N) = \langle \tilde{\phi}(k_1) \dots \tilde{\phi}(k_N) \rangle \tag{3.33}$$

are generated by functional differentiation of the generating functional

$$Z[\tilde{j}] = Z_0[0] \langle\, e^{-S_I[\tilde{\phi}] + \tilde{j}\tilde{\phi}} \,\rangle_0 . \tag{3.34}$$

Because of translational invariance, energy and momentum are globally conserved and these Green functions are proportional to the factor $\beta (2\pi)^3 \delta_{n,0} \delta(\mathbf{k})$. The delta-function appears as a consequence of us taking the continuum limit in momentum space. In the thermodynamic limit we formally have $\beta V \to (2\pi)^3 \delta(0)$.

Problem 3.3

Show that the free 2-point function is given by

$$\langle \tilde{\phi}(k_1) \tilde{\phi}(k_2) \rangle_0 = \beta (2\pi)^3 \tilde{\Delta}(k_1) \delta_{n_1 n_2} \delta(\mathbf{k}_1 - \mathbf{k}_2) \tag{3.35}$$

with (3.31) on account of translational invariance.

The Feynman rules may now be stated as follows:
  (i) Draw diagrams and determine symmetry factors as in vacuum field theory.
 (ii) Assign a propagator $\tilde{\Delta}(k)$ to each line and a factor $-\lambda$ to each vertex.
(iii) Conserve energy and momentum at each vertex according to the prescription: $\beta (2\pi)^3 \delta_{n,0} \delta(\mathbf{k})$; global energy and momentum conservation may be separated off by excluding one arbitrarily chosen vertex; this amounts to the omission of an overall factor $\beta (2\pi)^3 \delta(0) = \beta V$.
 (iv) Integrate and sum over all internal momenta and energies according to $\beta^{-1} \sum \int d^3 k / (2\pi)^3$.

By comparison with the Feynman rules of the vacuum theory [PS95], it is seen that the thermal Green functions may be obtained by making the following substitutions in the Minkowski-space Feynman integrals:

$$\tilde{G}(k_1, \dots, k_N) \quad \to \quad (-i)^N \tilde{G}^E(k_1, \dots, k_N) , \tag{3.36}$$

$$\int \frac{d^4 k}{(2\pi)^4 i} \quad \to \quad \beta^{-1} \sum_n \int \frac{d^3 k}{(2\pi)^3} , \tag{3.37}$$

$$i (2\pi)^4 \delta(k) \quad \to \quad \beta (2\pi)^3 \delta_{n,0} \delta(\mathbf{k}) . \tag{3.38}$$

The first prescription takes care of a factor of $(-i)$ for each external propagator and the last prescription ensures that effectively at each vertex a factor i disappears.

The same substitution rules apply in more complicated cases such as a SU(N) gauge theory with fermions. The first two substitution rules together imply that propagators loose a factor $(-i)$. It follows that for spin-$\frac{1}{2}$ fields the propagator becomes

$$\tilde{S}(k) = \frac{1}{m - \gamma \cdot k} \tag{3.39}$$

and that for (covariant) gauge and ghost fields the Matsubara propagators are

$$\tilde{D}^{ab}_{\mu\nu}(k) = \delta^{ab}\left[\frac{g_{\mu\nu}}{k^2} - (1 - \lambda^{-1})\frac{k_\mu k_\nu}{k^4}\right] , \tag{3.40}$$

$$\tilde{D}^{ab}_{g}(k) = \frac{\delta^{ab}}{k^2} . \tag{3.41}$$

These expressions are the Fourier transforms of the corresponding Euclidean propagators. Because of the periodic boundary conditions, discussed in the preceding chapter, the Matsubara frequencies of the ghost propagator are even, like those of the gauge boson propagator.

In actual calculations the above rules are supplemented with the usual prescriptions such as a minus sign for every fermion loop and ghost loop. Moreover, one may take over the renormalization prescriptions of the vacuum theory to eliminate all ultraviolet (UV) divergencies of the corresponding thermal field theory. On the basis of the Feynman rules stated above, it is possible to prove that thermal Green functions are UV-finite if the theory has been renormalized at zero temperature. This general result, that extends to any field theory, could have been expected on physical grounds. Indeed, the UV-divergences arise from the singular short-distance behavior of the theory, which is quite unaffected by the presence of a heath bath. In other words, in the UV limit the temperature and the chemical potential can effectively treated as being zero. This can already been seen from the form of the real-time propagator (1.59) because the vacuum part is neatly separated from the thermal part and the thermodynamic parameters enter only through the distribution function. This true for any field theory, as can been seen from the general structure of the propagator (1.58). In the Matsubara formalism this feature is less obvious, although it is implied by the representation (1.63). Nevertheless the same theorem has been proved. In practice, a convenient way of regulating the perturbation theory is to continue dimensionally as in the vacuum theory.

Although it is sufficient to renormalize the theory at $T = 0, \mu = 0$ in order to have a finite theory at any T, μ, it may be physically convenient to choose a renormalization prescription at a given $T, \mu \neq 0$. The renormalized masses and coupling then become T, μ dependent. Because the renormalized theory is finite at any T, μ, quantities renormalized at different renormalization point differ by a finite renormalization. Indeed the whole machinery of the renormalization group can be extended to thermal field theory.

3.4   Thermal mass

Let us consider as an elementary, but important, example of a perturbative calculation, the 2-point thermal Green function

$$\langle e^{-S_I}\tilde{\phi}(k)\tilde{\phi}(-k)\rangle_{\text{con}} = \tilde{G}^{(2)}(k)\beta(2\pi)^3\delta(0) \ , \tag{3.42}$$

where we extracted the overall factor βV. Expanding to first order we get

$$\tilde{G}^{(2)}(k) = \tilde{\Delta}(k) - \tfrac{1}{2}\tilde{\Delta}(k)\Pi\tilde{\Delta}(k) \ , \tag{3.43}$$

with the first-order self-energy given by the one-loop tadpole diagram

$$\Pi \ = \ \begin{array}{c}\text{K}\end{array}$$

$$= \ \tfrac{1}{2}\lambda\Delta(x=0) = \tfrac{1}{2}\lambda T \sum_n \int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega_n^2 + \omega_k^2} \ . \tag{3.44}$$

To first order in λ we find that Π is independent of the external momentum k. With the summation formula (1.69) we obtain:

$$\Pi = \tfrac{1}{2}\lambda \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}[1 + 2n(\omega_k)] \ . \tag{3.45}$$

The $T = 0$ part of the self energy is the vacuum contribution which is quadratically UV-divergent. It may be renormalized as at zero temperature by adding a temperature independent counter term to the Lagrangian. In dimensional regularization the tadpole at $T = 0$ vanishes and may be ignored completely.

The remainder is a convergent integral

$$\Pi = \frac{\lambda}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{\omega_k} n(\omega_k) \ . \tag{3.46}$$

This result is referred to as the *thermal mass*. Indeed, in general the self energy is defined through the relation

$$[\tilde{G}^{(2)}(k)]^{-1} = [\tilde{\Delta}(k)]^{-1} + \tilde{\Pi}(k) \ . \tag{3.47}$$

To first order in λ we infer from (3.43) that $\tilde{\Pi}(k)$ is given by expression (3.45). To this order, Π can be interpreted as a simple correction to the mass squared:

$$[\tilde{G}^{(2)}(k)]^{-1} = \omega_n^2 + \mathbf{k}^2 + m^2 + \Pi \ . \tag{3.48}$$

The thermal mass may be analytically evaluated for $m = 0$ using the formula (6.91) which yields the simple result

$$\Pi = \frac{\lambda}{(2\pi)^2}T^2 \int_0^\infty dx \frac{x}{e^x - 1} = \frac{\lambda T^2}{24} \ . \tag{3.49}$$

This thermal mass is the effective mass that particles acquire through their continuous interaction with the heat bath. It is one of the most important results of thermal field theory. For finite mass $m \ll T$ some further analytic results may be obtained by considering the asymptotic expansion in $u = \beta m$ of the integral

$$L_{2\nu}(u) = \frac{1}{(2\nu - 1)!} \int_u^\infty dx (x^2 - u^2)^{\nu - \frac{1}{2}} [1 + 2n(x)] . \tag{3.50}$$

Two useful instances of this high-temperature expansion are:

$$
\begin{align}
L_2(u) &= \tfrac{1}{3}\pi^2 - \pi u - \tfrac{1}{2}u^2 \log u + \mathcal{O}(u^2) , \tag{3.51} \\
L_4(u) &= \tfrac{1}{45}\pi^4 - \tfrac{1}{12}\pi^2 u^2 + \tfrac{1}{6}\pi u^3 + \mathcal{O}(u^4) . \tag{3.52}
\end{align}
$$

The first one gives the asymptotic result

$$\Pi = \frac{1}{24}\lambda T^2 \left[1 - \frac{3}{\pi}\frac{m}{T} - \frac{3}{2\pi^2}\log\frac{m}{T} + \dots \right] , \tag{3.53}$$

with the leading term proportional to the temperature squared and the sub-leading term of order m/T.

The next contribution in λ to the thermal mass is actually divergent when $m = 0$. This is an IR-divergence that has nothing to do with the UV divergences of the theory at $T = 0$. This new divergence at $T \neq 0$ is due to the fact that the boson develops a dynamically generated thermal mass of order gT (for notational convenience we write $g^2 = \lambda/24$). The problem arises because we expand in terms of a propagator that has zero mass to begin with. For example consider the 2-loop diagram

$$\Pi_2 = \qquad . \tag{3.54}$$

It is easy to show that the dominant IR-divergence of this diagram is

$$\Pi_2 \sim g^4 T^3 \int \frac{dk}{k^2} . \tag{3.55}$$

The divergence becomes more severe at each succeeding order.

Problem 3.4

Show that at order λ^N the dominant IR-divergent diagram is a daisy diagram consisting of one-loop with $N - 1$ self energy insertions $g^2 T^2$.
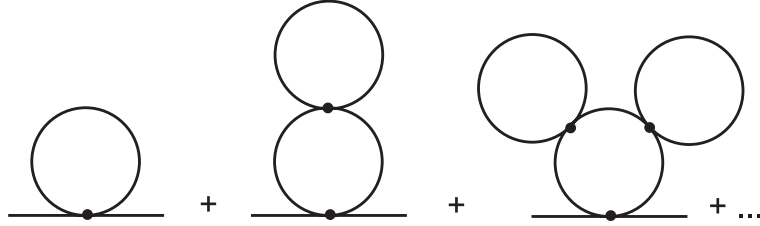
Figure 3.1: Daisy diagrams

This means that we cannot limit ourselves to naive perturbation theory, but must use a resummation. A further analysis shows that the symmetry factors are such that the insertion of a thermal mass can be done iteratively on all internal lines. All these daisy diagrams can be resummed into the self-consistent gap equation

$$M_\beta^2 = \tfrac{1}{2}\lambda T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_n^2 + \mathbf{k}^2 + M_\beta^2} \ . \tag{3.56}$$

In other words, we replace the massless propagator by one in which already a thermal mass M_β occurs. With the help of (3.50) this equation may be expressed as

$$u^2 = \frac{\lambda}{8\pi^2} L_2(u) \ . \tag{3.57}$$

The expansion in powers of λ is obtained from the asymptotic result (3.51)

$$u^2 = g^2 - \frac{3}{\pi} g^2 u \ , \tag{3.58}$$

from which we deduce:

$$M_\beta^2 = g^2 T^2 \left[1 - \frac{3}{\pi} g + \ldots \right] \ . \tag{3.59}$$

We see that the next term in M_β^2 is not of order g^4 as we would expect from naive perturbation theory, but instead of order g^3, that is, a non-perturbative correction. This reflects the breakdown of naive perturbation theory due to IR-divergences. Fortunately, the correction is small if the coupling constant g is small. This seems to imply that perturbation theory is reliable despite the IR-divergences, provided that leading terms are obtained with an effective propagator which includes the thermal mass (3.49). This mass can become rather large for high temperatures and the perturbation theory in terms of the zero-temperature mass breaks down.

The same feature also appears in the perturbative expansion of the partition function (2.59) when we include a thermal mass. Let us evaluate the partition function by separating off the massless part

$$
\begin{aligned}
\Omega &= \tfrac{1}{2} \sum_n \int \frac{d^3k}{(2\pi)^3} \left[\log(\omega_n^2 + \mathbf{k}^2) + \log\left(1 + \frac{M_\beta^2}{\omega_n^2 + \mathbf{k}^2}\right) \right] \\
&= \Omega_0 + \Omega_{\text{ex}} + \Omega_{\text{ring}} \ .
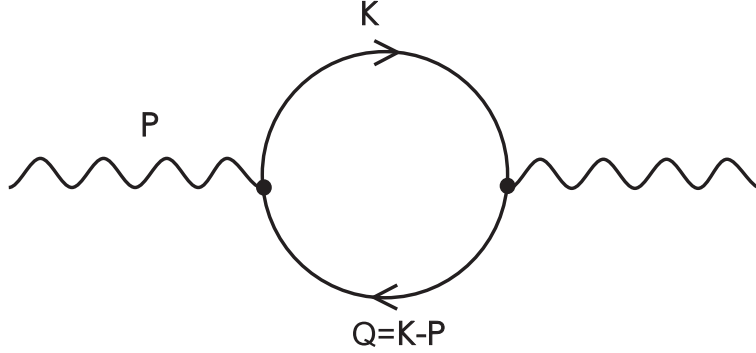\end{aligned}
\tag{3.60}
$$

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Figure 3.2: Photon self energy

The first term corresponds to a free massless Bose gas, and the third term is the so called ring contribution

$$\Omega_{\text{ring}} = \tfrac{1}{2} \sum_n \int \frac{d^3 k}{(2\pi)^3} \left[\log \left(1 + \frac{M_\beta^2}{\omega_n^2 + \mathbf{k}^2} \right) - \frac{M_\beta^2}{\omega_n^2 + \mathbf{k}^2} \right] . \qquad (3.61)$$

The subtracted term is the so-called exchange term Ω_{ex} which corresponds to the lowest order term (3.20). This term is subtracted because it is IR-finite. On the other hand, the ring term can be seen as a resummation of the IR-divergences that come from the static mode $n = 0$. In contrast, for $n \neq 0$ the Matsubara frequencies $2\pi n T$ act as a mass in the propagator and provide a cutoff in the momentum integrals. If one restricts oneself to the $n = 0$ term in Ω_{ring}, the integral is easily evaluated after an integration by parts on the logarithm.

Problem 3.5

Evaluate Ω_{ring} for $n = 0$ to obtain the leading order

$$\Omega_{\text{ring}} \simeq -\frac{g^3 T^3}{12\pi} \qquad (3.62)$$

showing that the IR divergences lead to a g^3 dependence in the coupling constant.

3.5 Photon self energy

The self energy of a gauge boson in the one-loop approximation is an important example for the application of perturbation theory at finite temperature. For simplicity we first treat the photon self energy. We consider a thermal system of relativistic electrons, positrons, and photons with a net charge zero, i.e. with a vanishing chemical potential. The transversality of the polarization tensor $p_\mu \Pi^{\mu\nu}(p)$, implies that only two components

are independent, for which we choose the longitudinal and the transverse components

$$\Pi_L(p) = \Pi_{00} , \tag{3.63}$$

$$\Pi_T(p) = \tfrac{1}{2}\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right)\Pi_{ij}(p) . \tag{3.64}$$

The starting point is the one-loop expression as it follows from the finite temperature Feynman rules

$$\Pi_{\mu\nu}(p) = -e^2 T \sum_n \int \frac{d^3k}{(2\pi)^3} \, \mathrm{tr}[\gamma_\mu \tilde{S}(k-p)\gamma_\nu \tilde{S}(k)] . \tag{3.65}$$

Since the loop consists of fermion lines, the propagator is the Dirac-propagator $\tilde{S}(k) = (\gamma \cdot k + m)\tilde{\Delta}(k)$, with $k_0 = i(2n+1)\pi T$ discrete odd Matsubara frequencies; the rules include a minus sign for the loop. The trace over the γ-matrices gives

$$\mathrm{tr}\left[\gamma^\mu \, q \cdot \gamma \, \gamma^\nu \, k \cdot \gamma\right] = 4\left[q^\mu k^\nu + k^\mu q^\nu - g^{\mu\nu}(q \cdot k)\right] , \tag{3.66}$$

with the abbreviation $q = p - k$. Restricting ourselves to the longitudinal component and making use of the mixed representation for the fermionic Matsubara propagator

$$\Delta(\tau, \mathbf{k}) = \sum_{s=\pm 1} \frac{s}{2\omega_k}\left[1 - n_F(s\omega_k)\right]e^{-s\tau\omega_k} , \tag{3.67}$$

we find

$$\Pi_L(p) = -4e^2 T \int \frac{d^3k}{(2\pi)^3} \sum_{k_0} (k_0 q_0 + \mathbf{k}\cdot\mathbf{q}) \int_0^\beta d\tau d\tau' e^{k_0\tau + q_0\tau'} \Delta(\tau, \mathbf{k})\Delta(\tau', \mathbf{q}) , \tag{3.68}$$

where we neglected the fermion mass: $T \gg m_e$.

In order to evaluate the sum over k_0, we replace the term proportional to $k_0 q_0$ by derivatives with respect to τ, τ' and integrate by parts:

$$\begin{aligned}\Pi_L(p) = & -4e^2 T \int \frac{d^3k}{(2\pi)^3} \sum_{k_0} \int_0^\beta d\tau d\tau' e^{k_0(\tau-\tau')}e^{p_0\tau'} \\ & \times \left[\frac{d\Delta(\tau, \mathbf{k})}{d\tau}\frac{d\Delta(\tau', \mathbf{q})}{d\tau'} + \mathbf{k}\cdot\mathbf{q}\,\Delta(\tau, \mathbf{k})\Delta(\tau', \mathbf{q})\right] . \end{aligned} \tag{3.69}$$

We may perform the sum over k_0 with the help of the identity

$$T \sum_{n=-\infty}^{\infty} e^{i\omega_n(\tau-\tau')} = \delta(\tau - \tau') \tag{3.70}$$

and integrate over τ' afterwards. We then get:

$$\begin{aligned}\Pi_L(p) = & -e^2 \int \frac{d^3k}{(2\pi)^3} \int_0^\beta d\tau e^{p_0\tau} \\ & \times \sum_{r,s} \left[1 + rs\frac{\mathbf{k}\cdot\mathbf{q}}{\omega_k\omega_q}\right] n_F(s\omega_k)n_F(r\omega_q)\,e^{\tau(s\omega_k+r\omega_q)} . \end{aligned} \tag{3.71}$$

In this expression the external energy is still imaginary with discrete values $p_0 = i2nT$ as the photon is a boson, and all other energies real. The final integration over τ can be performed and yields

$$\int_0^\beta d\tau e^{\tau(p_0 + s\omega_k + r\omega_q)} = \frac{e^{\beta(s\omega_k + r\omega_q)} - 1}{p_0 + s\omega_k + r\omega_q} \ . \tag{3.72}$$

Then the result is

$$\begin{aligned}\Pi_L(p) &= -e^2 \int \frac{d^3k}{(2\pi)^3} \sum_{r,s} \left[1 + rs\frac{\mathbf{k} \cdot \mathbf{q}}{\omega_k \omega_q} \right] \\ &\times \ n_F(s\omega_k) n_F(r\omega_q) \frac{e^{\beta(s\omega_k + r\omega_q)} - 1}{p_0 + s\omega_k + r\omega_q} \ . \end{aligned} \tag{3.73}$$

Now that the summation over the internal energies has been done, we may continue the external variable analytically into the complex plane $p_0 \to z$ and to the real axis: $z \to p_0 + i\epsilon$, where p_0 is now the real valued energy of the photon. The ϵ-prescription gives the retarded self energy.

The photon self energy (3.73) cannot be evaluated further analytically. However, in the high-temperature limit, which is of the most interest, the integral can be simplified considerably. In the literature this is referred to as the "Hard Thermal Loop (HTL)" approximation. We have already assumed that the temperature is much larger than the electron mass: $\omega_k \sim |\mathbf{k}|$. Let us also assume that the external energy and momentum are small compared to the temperature: $T \gg p_0, |\mathbf{p}|$. On the other hand, the internal momentum is of order $|\mathbf{k}| \sim T$. This allows the expansion: $\omega_q = |\mathbf{p} - \mathbf{k}| = |\mathbf{k}| - \mathbf{p} \cdot \mathbf{v}, \mathbf{v} = \mathbf{k}/|\mathbf{k}|$. As a result we have:

$$\frac{\mathbf{k} \cdot \mathbf{q}}{\omega_k \omega_q} \simeq -1 \tag{3.74}$$

$$p_0 \pm (\omega_k - \omega_q) \simeq p_0 \pm \mathbf{p} \cdot \mathbf{v} \tag{3.75}$$

$$n_F(\omega_k) - n_F(\omega_q) \simeq \mathbf{p} \cdot \mathbf{v} \left. \frac{dn_F(\epsilon)}{d\epsilon} \right|_{\epsilon = |\mathbf{k}|} \ . \tag{3.76}$$

The first approximation leads to a cancelation of half of the terms in (3.73) and we get

$$\Pi_L(p) = 2e^2 \int \frac{d^3k}{(2\pi)^3} [n_F(\omega_k) - n_F(\omega_q)] \left(\frac{1}{p_0 + i\epsilon + \omega_k - \omega_q} - \frac{1}{p_0 + i\epsilon - \omega_k + \omega_q} \right) \ . \tag{3.77}$$

To obtain this form we also used the identity for the Fermi-Dirac distribution function

$$\frac{n_F(\epsilon)}{1 - n_F(\epsilon)} = e^{-\beta\epsilon} \ . \tag{3.78}$$

Next we use the second two approximations. We then see that the angular and radial integrations become completely decoupled. The radial integration is performed with the help of

$$\int_0^\infty dk k^2 \frac{dn_F(k)}{dk} = -\frac{\pi^2 T^2}{6} \ . \tag{3.79}$$

This then leads to the high-temperature expression for the longitudinal photon self energy

$$\Pi_L(p) = 3\omega_{\rm pl}^2 \int \frac{d\Omega}{4\pi} \left(\frac{p_0}{p_0 - \mathbf{v} \cdot \mathbf{p} + i\epsilon} - 1 \right) , \tag{3.80}$$

where $d\Omega$ indicates the angular integration over the direction of $\mathbf{v}$. The plasmon frequency $\omega_{\rm pl}$, which in the high-T limit has the value $\omega_{\rm pl} = eT/3$, may be considered as the thermal photon mass generated by the interaction of the photons with the electrons and positrons.

Expression (3.80) is the "Hard Thermal Loop (HTL)" approximation for the photon self energy. Its basic feature is the proportionality to T^2 as it can be shown that the terms neglected are all of lower order in the temperature; the next term is proportional to T.

Problem 3.6

Perform the angular integration and show that the result can be expressed as

$$\Pi_L(p) = -\frac{e^2 T^2}{3} \left[1 - \frac{p_0}{|\mathbf{p}|} Q\left(\frac{p_0}{|\mathbf{p}|} \right) \right] , \tag{3.81}$$

in terms of the Legendre function

$$Q(x) = \tfrac{1}{2} \log \frac{x+1}{x-1} \tag{3.82}$$

defined in the complex plane cut from -1 to 1.

The above result makes explicit that the HTL photon self energy is of order T^2 and that it has an imaginary part for space-like momenta $|\mathbf{p}|^2 > p_0^2$:

$$\log \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|} = \log \left| \frac{p_0 + |\mathbf{p}|}{p_0 - |\mathbf{p}|} \right| - i\pi\theta(|\mathbf{p}|^2 - p_0^2) . \tag{3.83}$$

The resulting damping stems from the scattering of electron and positrons with momenta of order T on low-momentum photons. In plasma physics this phenomenon is known as *Landau damping* and it plays an important role in calculations of the energy loss or the viscosity. We also note that in the static limit $p_0 = 0$ the self energy is independent of $|\mathbf{p}|$ and reduces to $\Pi(0) = -3\omega_{\rm pl}^2$. The static limit is related to the screening of static fields and $m_D^2 = 3\omega_{\rm pl}^2$ is the Debye mass.

Chapter 4

Response Theory and Applications

When considering the properties of many-body systems, one is often interested in the response of the system to an external disturbance. A simple example is provided by the susceptibility of a system containing particles with a magnetic moment: one applies a magnetic field and measures the resulting magnetic moment. Or one wants to study the propagation of gauge bosons and fermions in a QED or QCD plasma by setting up some disturbance. In all these situations one employs a probe which disturbs the system only slightly from equilibrium and measures the linear time-dependent response which can be Fourier analyzed into what is called a frequency dependent susceptibility.

In linear response theory the susceptibilities are found in terms of time-dependent retarded correlation functions. The collective modes appear as poles of these correlation functions. The imaginary part of the pole describes the decay time of collective excitations, which usually is finite in a thermal system, and the real part the dispersion law giving the energy as a function of momentum. However, even if no well-defined external disturbance exists, as in the case of particle production and damping, it is still possible to express the production rate, the damping coefficient and other thermal transport coefficients, such as the viscosity, in terms of correlation functions. This indicates that retarded time-correlation functions play an extremely important role in non-equilibrium statistical mechanics. That goes beyond the pure linear response theory in which they originate.

4.1   Response theory

We begin with a short review of response theory. Consider a system in equilibrium with a time-independent Hamiltonian $\hat{H}$. Suppose now that at some initial time $t = t_i$ the system is perturbed by turning on an additional time-dependent Hamiltonian of the form

$$\hat{V}(t) = - \int d^3x \, \hat{a}(\mathbf{x}) \, h(t, \mathbf{x}) \,, \qquad (4.1)$$

where $\hat{a}(\mathbf{x})$ is a hermitian operator corresponding to some observable local density (e.g. the charge density or energy density), and $h(t, \mathbf{x})$ an external c-number field. We assume

that $\hat{V}(t)$ vanishes for times before t_i, and that at this time the system is in thermal equilibrium.

Generally what one wishes to calculate is the average $a(t, \mathbf{x}) \equiv \langle \hat{a}(t, \mathbf{x}) \rangle$ at a time $t > t_i$, with the equilibrium value at $t = t_i$ presumed to be known. To find the time development of $a(t, \mathbf{x})$, we first consider the Schrödinger picture in which $\hat{a}(\mathbf{x}) = \hat{a}(t_i, \mathbf{x})$ is time independent, and the exact state vector satisfies the Schrödinger equation

$$i\partial_t |\psi\rangle_t = [\hat{H} + \hat{V}(t)]|\psi\rangle_t . \tag{4.2}$$

We shall seek the solution in the form

$$|\psi\rangle_t = e^{-i\hat{H}(t-t_i)}\hat{U}(t, t_i) |\psi\rangle_t , \tag{4.3}$$

where $\hat{H}$ is the Hamiltonian of the unperturbed system, and where the unitary operator $\hat{U}(t, t_i)$ obeys the initial condition $\hat{U}(t_i, t_i) = 1$. Substitution of equation (4.3) into (4.2) yields the evolution equation for $\hat{U}$:

$$i\partial_t \hat{U}(t, t_i) = \hat{V}_H(t)\hat{U}(t, t_i) . \tag{4.4}$$

Here $\hat{V}_H(t)$ is the external Hamiltonian in the ordinary Heisenberg picture defined with respect to time t_i:

$$\hat{V}_H(t) \quad \equiv \quad e^{i\hat{H}(t-t_i)}\hat{V}(t)e^{-i\hat{H}(t-t_i)} \tag{4.5}$$

$$= \quad -\int d^3x \, \hat{a}_H(t, \mathbf{x}) \, h(t, \mathbf{x}) . \tag{4.6}$$

The formal solution of the evolution equation (4.4) is

$$\hat{U}(t, t_i) = T \exp -i \int_{t_i}^{t} dt' \, \hat{V}_H(t') \tag{4.7}$$

with T the time-ordering operator.

If we now bring the full interaction to the operators by defining

$$\hat{a}(t, \mathbf{x}) = \hat{U}(t_i, t)\hat{a}_H(t, \mathbf{x})\hat{U}(t, t_i) , \tag{4.8}$$

the state vector becomes constant in time and describes the initial state which by assumption is the equilibrium state. Therefore we may calculate the expectation value $\langle \hat{a}(t, \mathbf{x}) \rangle$ by taking the equilibrium average of the above equation:

$$\langle \hat{a}(t, \mathbf{x}) \rangle = \frac{1}{Z}\text{Tr}e^{-\beta\hat{H}}\hat{U}(t_i, t)\hat{a}_H(t, \mathbf{x})\hat{U}(t, t_i) , \tag{4.9}$$

where all operators at the right-hand side are in the Heisenberg picture. The time-order implied by this expression is one in which the time runs from the initial time t_i to the actual time t, and then back again to t_i. In fact one can let the time run up to some arbitrary final time $t_f > t$ and then back again. Finally, the Boltzmann factor then
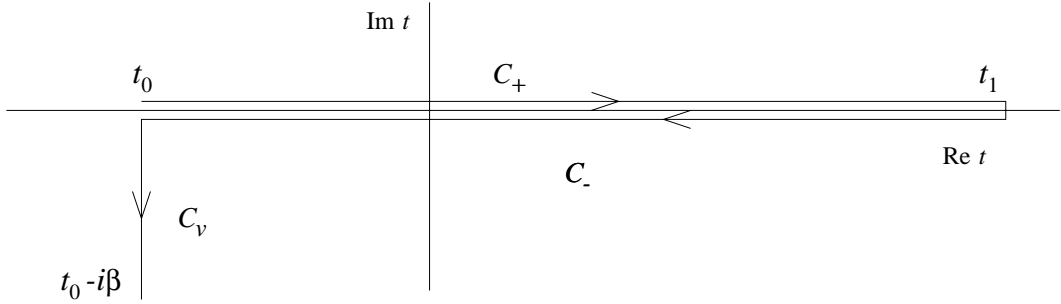
Figure 4.1: Thermal field time-contour in the complex t plane

propagates the time from t_i to $t_i - i\beta$. The successive time-contours may be joined to form a single contour C, running up and down the real axis from and to t_i and then down to $t_i - i\beta$. Results should be independent of the initial and final times and in real-time thermal field theory one usually takes the limit $t_i \to -\infty$ and $t_f \to \infty$. Note that in the trace the Boltzmann factor could be moved to the right; this would yield a contour with the imaginary part at the beginning. Other contours are also possible, and it is mainly a matter of taste which one to prefer.

This situation is quite different from the vacuum theory. There Green functions are usually defined by reducing the transition element to an in/out state amplitude $\langle \text{out}|\text{in}\rangle$. The corresponding path integral has then a time contour from $-\infty$ to ∞. However in statistical mechanics, because expectation values are calculated by a trace, one always has to come back to the initial time.

4.2 Linear response

Let us now restrict ourselves to terms linear in the perturbing field $h(t, \mathbf{x})$ (Born approximation). Then the solution of equation (4.4) is

$$\hat{U}(t, t_i) = 1 - i \int_{t_i}^{t} dt' \hat{V}_H(t') \ . \tag{4.10}$$

Hence, the first-order change in $\langle \hat{a}(t, \mathbf{x})\rangle$ arising from an external perturbation can be expressed as

$$\langle \hat{a}(t, \mathbf{x})\rangle = \langle \hat{a}_H(t, \mathbf{x})\rangle + i \int_{t_i}^{\infty} dt' \, \theta(t - t')\langle \, [\hat{V}_H(t'), \hat{a}_H(t, \mathbf{x})] \, \rangle \ , \tag{4.11}$$

in terms of the retarded commutator of the perturbation and the operator $\hat{a}_H(t, \mathbf{x})$ in the Heisenberg picture of the interacting but unperturbed system. Inserting (4.6) we find that the linear response of the system induced by the external field to linear order is given by

$$\delta\langle a(x)\rangle = i \int_{t_i}^{\infty} dt' \int d^3x' \theta(t - t')\langle \, [\hat{a}(x), \hat{a}(x')] \, \rangle \, h(x') \ , \tag{4.12}$$

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where the thermal average of the retarded commutator is called the retarded response function.

We apply this to the important case of the scalar field:

$$\hat{V}(t) = - \int d^3x \, \hat{\phi}(\mathbf{x}) j(t, \mathbf{x}) , \qquad (4.13)$$

where $j(x) = j(t, \mathbf{x})$ is the external source and $\hat{\phi}(\mathbf{x})$ the field operator in the Schrödinger picture. We then obtain

$$\delta \langle \hat{\phi}(x) \rangle = i \int d^4x \, G^R(x - x') \, j(x') , \qquad (4.14)$$

where

$$G^R(x - x') = \theta(t - t') \langle \, [\hat{\phi}(x), \hat{\phi}(x')] \, \rangle \qquad (4.15)$$

is the retarded Green function which describes the causal behavior of the response. We could replace the lower limit of the time integral by $-\infty$, since the external field vanishes for $t < t_i$. Furthermore, we may forget that the Heisenberg picture was defined with respect to the time t_i, on the strength of the translational invariance of equilibrium correlation functions.

Equations (4.14) with (4.15) is the fundamental result of linear response theory. It shows that the response function is the averaged commutator of two Heisenberg operators at different times, i.e. a time correlation function. It represents the delay between the applied field and the induced current. If the system can respond instantaneously then the response function is a δ-function in time. Ordinarily, however, the system lags behind and the response function is a monotonically decaying function of time.

Since we consider a system in equilibrium, the response function is translationally invariant in space and time and we can define the Fourier transform of the commutator

$$\langle \, [\hat{\phi}(x), \hat{\phi}(x')] \, \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x - x')} \rho(k) , \qquad (4.16)$$

which is called the spectral density. We know that $\rho(k)$ is real, since it is a commutator of hermitian operators, an odd function of k_0, and that it only depends on $|\mathbf{k}|$.

Problem 4.1

Show that the Fourier transform of the retarded response function has the spectral form

$$\tilde{G}^R(k) = \int_{-\infty}^{\infty} \frac{dk_0'}{2\pi} \frac{\rho(k_0', \mathbf{k})}{k_0 - k_0' + i\epsilon} \qquad (4.17)$$

and compare with the free case (1.74).

In terms of this retarded response function the linear response (4.14) may be written as

$$\delta \langle \tilde{\phi}(k) \rangle = \tilde{G}^R(k) \tilde{j}(k) , \qquad (4.18)$$

which shows that the system responds at the same wave vector and frequency as the perturbation. The quantity $\tilde{G}^R(k)$ is therefore a generalization of the complex dynamical magnetic susceptibility as it is usually defined in electromagnetism. Its imaginary part describes decay and its real part, which is connected to the imaginary part by a Kramers-Kronig relation, dispersion.

We consider one last quantity, namely the complex response function

$$G(z, \mathbf{k}) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\rho(k)}{k_0 - z} \, , \tag{4.19}$$

which is an analytic function of the complex frequency variable z as long as Im $z \neq 0$. The retarded response function, i.e. the physical response, is given by the limit as z approaches the real axis from above. This complex response function looks exactly like the analytic propagator (1.79) introduced in chapter 1, except that $\rho(k)$ is now the spectral density of the interacting system. In the lower half plane $G(z, \mathbf{k})$ corresponds to the Fourier-Laplace transform of the response function for negative times, that is, the advanced response function.

Problem 4.2

a. *Show that the boundary value in the lower half plane is equal to the advanced response function*

$$\tilde{G}^A(k) = G(k_0 - i\epsilon, \mathbf{k}) \, . \tag{4.20}$$

b. *Identify the spectral density with the discontinuity*

$$\tilde{G}^R(k) - \tilde{G}^A(k) = i\rho(k) \tag{4.21}$$

across the branch cut.

The above described derivation of linear response theory seems to be completely rigorous since it only uses first-order perturbation theory. Nevertheless, this simple argument hides a fallacy, as has been argued by van Kampen (1971). The point he makes is the following: it is certainly true that the macroscopic response of a system is often linear. However, it is false to assume that this linearity is the result of a linearity of the equations of motion on the microscopic level. The linearity of macroscopic phenomena is a much more subtle effect and involves the fundamental ideas of statistical mechanics. In contrast, the simple derivation of linear response given above is completely dynamical and makes no use at all of statistical arguments. The question why linear-response theory still provides a correct description in spite of a derivation which is so obviously wrong, is very difficult to answer in detail. The general idea is that the precise details of the motion of the particles are very quickly forgotten on the scale of the external disturbance. What results is an average behavior which is simulated by the linear approximation.

4.3 Collective excitations

The properties of particles are modified when they propagate in a medium, as they become dressed by their interactions. For example, they may acquire an effective mass which is different from the vacuum mass. Generally, one speaks of collective modes in the case of bosons and of quasi-particles in the case of fermions. The quasi-particle concept is of fundamental importance in analyzing many-particle systems. It often allows one to replace a complicated system of strongly interacting particles by an equivalent system of weakly interacting quasi-particles. In some case the quasiparticles are easily identified with elementary particles whose properties are only slightly modified by their interactions with the medium. On the other hand, in other cases the collective modes have no vacuum analogon and are a purely collective effect propagating through the medium.

Collective modes (quasiparticles) are characterized by a dispersion relation $\omega(\mathbf{k})$ giving their energy as a function of momentum. Usually their life-time is not infinite and another relevant quantity is the damping rate $\gamma(\mathbf{k})$. Collective modes are identified as poles of the retarded propagator in the complex energy plane. The bare retarded propagator $\tilde{D}^R = 1/(k^2 - m^2 + i\mathrm{sign}\,k_0\epsilon)$ describes the dispersion relation of a non-interacting scalar particle. One may note that both poles $k_0 = \pm\omega_k - i\epsilon$ are below the real axis.

In the interacting case, the dispersion relation usually will have an imaginary part. As an example, we consider a scalar field. Let us assume that the interacting retarded Green function is given as the boundary value of the analytic function

$$\begin{aligned} \tilde{G}^R(k) &= G(k_0 + i\epsilon, \mathbf{k}) \\ G(z, \mathbf{k}) &= \frac{1}{z^2 - \omega_k^2 - \Pi(z, \mathbf{k})} \ , \end{aligned} \tag{4.22}$$

which has no poles in the upper half plane. In general the the frequency will have a real and an imaginary part. Defining $\mathrm{Re}\,z = \omega$ and the damping rate $\gamma = -\mathrm{Im}\,z$, we have for a plane wave $\exp(-izt) = \exp(-i\omega t)\,\exp(-\gamma t)$. The dispersion relation is given by the solution of the equation

$$(\omega - i\gamma)^2 - \omega_k^2 - \mathrm{Re}\,\Pi(\omega - i\gamma, \mathbf{k}) - i\mathrm{Im}\,\Pi(\omega - i\gamma, \mathbf{k}) = 0. \tag{4.23}$$

In the case of no overdamping, $\gamma \ll \omega$, we get

$$\begin{aligned} \omega^2 - \omega_k^2 - \mathrm{Re}\,\Pi^R(\omega, \mathbf{k}) &= 0 \ , \tag{4.24} \\ 2i\omega\gamma + i\,\mathrm{Im}\,\Pi^R(\omega, \mathbf{k}) &= 0 \ , \tag{4.25} \end{aligned}$$

where $\Pi^R(\omega, \mathbf{k}) = \Pi(\omega + i\epsilon, \mathbf{k})$. From the first relation we get the dispersion relation $\omega = \omega(\mathbf{k})$, and from the second one

$$\gamma = -\frac{1}{2\omega_k}\,\mathrm{Im}\,\Pi^R(\omega(\mathbf{k}), \mathbf{k}) \ . \tag{4.26}$$

If we find such a pole with a small damping rate γ, then we say that there is a collective mode (quasi-particle) with energy $\hbar\omega(\mathbf{k})$ and life time γ^{-1}. If there are several poles, the

dominant term for large t arises from the pole closest to the real axis, because it has the least damping.

Alternatively, we may derive the dispersion relation and the life-time from the spectral density which is obtained by calculating:

$$\begin{aligned}
\rho(k) &= -\frac{1}{\pi} \mathrm{Im} \frac{1}{k_0^2 - \omega_k^2 - \mathrm{Re}\,\Pi^R(k) - i\mathrm{Im}\,\Pi^R(k)} \ , \\
&= \frac{1}{\pi} \frac{2\mathrm{Im}\,\Pi^R(k)}{[k_0^2 - \omega_k^2 - \mathrm{Re}\,\Pi^R(k)]^2 + [\mathrm{Im}\,\Pi^R(k)]^2}
\end{aligned} \tag{4.27}$$

The spectral function has the Breit-Wigner form describing excitations with finite width. If the self energy is a slowly varying function of energy we can think of $\mathrm{Im}\,\Pi^R(k)$ as the energy and momentum dependent life-time of a single particle state, and regard $\mathrm{Re}\,\Pi^R(k)$ as the energy gained by the particle through its interaction with the surrounding medium. To see this more explicitly, we first look for the solution of the dispersion relation:

$$k_0^2 - \omega_k^2 - \mathrm{Re}\,\Pi^R(k) = 0 \ . \tag{4.28}$$

If such a solution $\omega(\mathbf{k})$ exists, it defines the position of the peak. For energies not too far from $\omega(\mathbf{k})$, we can expand the slowly varying functions around

$$\begin{aligned}
\mathrm{Re}\,\Pi^R(k) &= M^2(\mathbf{k}) + [k_0 - \omega(\mathbf{k})]M'(\mathbf{k}) + \cdots \tag{4.29} \\
\mathrm{Im}\,\Pi^R(k) &= -\Gamma(\mathbf{k}) + \cdots \ . \tag{4.30}
\end{aligned}$$

To first order we obtain in this way

$$\rho(k) \simeq -\frac{1}{\pi} \mathrm{Im} \frac{Z(\mathbf{k})}{k_0^2 - \omega(\mathbf{k})^2 + i\gamma(\mathbf{k})} \ , \tag{4.31}$$

where we introduced the damping constant $\gamma(\mathbf{k}) = Z(\mathbf{k})\Gamma(\mathbf{k})$ and the renormalization factor $Z(\mathbf{k}) = 1/[1 - M'(\mathbf{k})]$. If the damping is zero, the spectral density reduces to delta-function at the energy $\omega(\mathbf{k})$. In the more general case of non-vanishing, but small, imaginary part, the spectral density has a Lorentzian line-shape representing a dynamical quasiparticle

Now let us take as an example the longitudinal high-temperature photon propagator calculated in section 3.6. The dispersion relation of the collective photon mode in the relativistic electron-positron plasma follows from the poles of the effective propagator

$$[\tilde{G}^R(k)]^{-1} = k^2 + 3\omega_{\mathrm{pl}}^2 \left(1 - \frac{k_0}{2|\mathbf{k}|} \log \frac{k_0 + |\mathbf{k}|}{k_0 - |\mathbf{k}|}\right) = 0 \ . \tag{4.32}$$
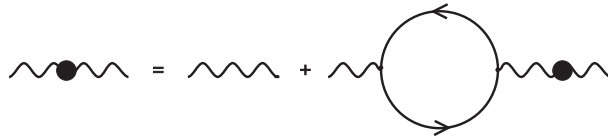

Figure 4.2: Effective photon propagator

In the static limit $k_0 \to 0$, the effective propagator reduces to $1/(k^2 + 3\omega_{\mathrm{pl}}^2)$. Thus, the long-range interaction is screened in the IR limit by the Debye mass $m_D = \sqrt{3}\omega_{\mathrm{pl}}$ owing to the presence of electric charges in the medium.

The equation (4.32) cannot be solved analytically. In the limits, $|\mathbf{k}| \to 0$ and $|\mathbf{k}| \to \infty$, however, there are simple solutions:

$$\omega(|\mathbf{k}| \to 0) = \omega_{\mathrm{pl}} + \frac{3}{10} \frac{|\mathbf{k}|^2}{\omega_{\mathrm{pl}}} , \tag{4.33}$$

$$\omega(|\mathbf{k}| \to \infty) = |\mathbf{k}| . \tag{4.34}$$

In the last limit the dispersion relation of the bare photon is recovered. In the limit $|\mathbf{k}| \to 0$, on the other hand, the dispersion relation describes a non-relativistic particle with mass ω_{pl}. This particle corresponds to a collective mode called the plasmon. The dispersion relation lies above the light cone, $\omega > |\mathbf{k}|$, for all values of $|\mathbf{k}|$. Therefore, the imaginary part of the self-energy vanishes and plasma oscillations are not damped in the high-temperature approximation, that is, there is no Landau damping causing dissipation of energy from the plasma wave in a heat bath. In a QGP there are similar collective gluon modes.

4.4 Production rate

An important application of thermal field theory is the calculation of damping and production rates of particles at high-T in gauge theories. The recent experimental interest in the Quark Gluon Plasma (QGP) has been a major stimulus for a detailed study of production rates and energy losses of various processes. For example, electromagnetic probes (photons, dileptons) have been proposed as promising signatures of the formation of QGP. When these particles are produced in relativistic heavy-ion collisions, they leave the fireball without further interactions, thus providing a direct probe for the strongly interacting system.
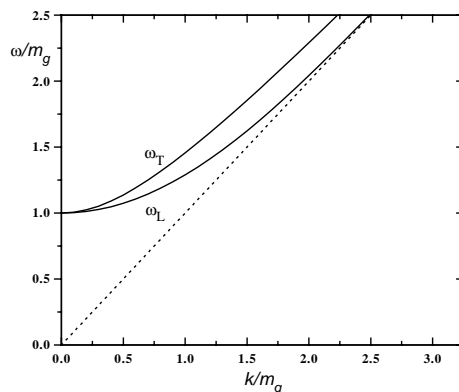

Figure 4.3: Dispersion relation of transverse (upper branch) and longitudinal (lower brach) plasma waves

Damping rates and production rates are related to each other by the principle of detailed balance. Since the photon damping rate to lowest order in naive perturbation theory is logarithmically IR-divergent for a vanishing quark mass, we consider here as a simpler example, the thermal production of a massive particle, which interacts only weakly with the particles of the heat bath. When it is produced it escapes and never has enough interactions with the heat bath to thermalize. We assume the particles of the heat bath to be scalar bosons ϕ, which we take massless. We denote the heavy scalar particle as Φ, and assume that the production occurs through the reaction $\phi + \phi \to \Phi$, which is governed by the interaction $\lambda \Phi(x)\phi(x)^2$, with a small coupling constant λ. The scalar fields ϕ may interact strongly among themselves, but we do not need specify this interaction. We could think of this as a model for the production of a heavy virtual photons in a QGP plasma.

In the description of the production process we assume the initial state of the system to be a stationary state $|\varphi_i\rangle$ with energy E_i. At $t = t_i$ the system is disturbed by turning on the additional interaction described by the time-dependent Hamiltonian

$$\hat{V}_H(t) = \lambda \int d^3x \, \hat{\Phi}(x)\hat{\phi}(x)^2 \; . \tag{4.35}$$

The amplitude for finding the system at time t_f in the stationary state $|\varphi_f\rangle$ corresponding to the energy E_f is then given by

$$
\begin{aligned}
\langle \varphi_f | \varphi_i \rangle &= \langle \varphi_f | e^{i\hat{H}(t-t_i)} \hat{U}(t,t_i) | \varphi_i \rangle \\
&= \langle \varphi_f | \hat{U}(t,t_i) | \varphi_i \rangle e^{-iE_f(t-t_i)} \; .
\end{aligned}
\tag{4.36}
$$

Hence the increase per unit time of the probability that a system initially in the state $|\varphi_i\rangle$ will be found at time t in the state $|\varphi_f\rangle$ is

$$w_{fi} = \frac{d}{dt} \left| \langle \varphi_f | \hat{U}(t,t_i) | \varphi_i \rangle \right|^2 \; . \tag{4.37}$$

Making use of the equation of motion (4.4), we note that this may be written as

$$w_{fi} = 2 \, \text{Im} \, \langle \varphi_f | \hat{V}_H \hat{U}(t,t_i) | \varphi_i \rangle \langle \varphi_i | \hat{U}(t_f,t) | \varphi_f \rangle \; . \tag{4.38}$$

If we assume that $|\varphi_i\rangle \neq |\varphi_f\rangle$ and that the Born approximation is sufficiently accurate, we obtain

$$w_{fi} = 2i \, \text{Im} \, \langle \varphi_f | \hat{V} | \varphi_i \rangle \int_{t_i}^{t} dt' e^{i(E_i - E_f)(t'-t)} \langle \varphi_i | \hat{V} | \varphi_f \rangle \; . \tag{4.39}$$

Now we take the lower limit as $t_i = -\infty$ and suppose that the perturbation is being turned on adiabatically $\hat{V} \to \hat{V}e^{\epsilon t}$:

$$
\begin{aligned}
w_{fi} &= 2 \, \text{Im} \frac{1}{E_i - E_f - i\epsilon} \left| \langle \varphi_f | \hat{V} | \varphi_i \rangle \right|^2 \\
&= 2\pi\delta(E_i - E_f) \left| \langle \varphi_f | \hat{V} | \varphi_i \rangle \right|^2 \; .
\end{aligned}
\tag{4.40}
$$

This result, which is called Fermi's Golden Rule, is independent of time and gives the transition probability per unit time from the initial to the final state.

We apply the golden rule to the example of the production of Φ particles. The matrix element for a Φ-particle to be produced with energy and momentum q is

$$\langle \varphi_f | \hat{V} | \varphi_i \rangle = \lambda \int d^3x \langle q | \hat{\Phi}(\mathbf{x}) | 0 \rangle \langle f | \hat{\phi}(\mathbf{x})^2 | i \rangle \ , \tag{4.41}$$

where $|i\rangle$ and $|f\rangle$ refer to the states of the ϕ-particles. The first matrix element is easily calculated and gives

$$\langle q | \hat{\Phi}(\mathbf{x}) | 0 \rangle = \frac{1}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{x}} \ . \tag{4.42}$$

To derive this we have used the standard decomposition of the operator $\hat{\Phi}$ in plane waves. The matrix element (4.41) so becomes

$$\langle \varphi_f | \hat{V} | \varphi_i \rangle = \frac{\lambda}{(2\pi)^3} \int d^3x \, e^{-i\mathbf{q}\cdot\mathbf{x}} \langle f | \hat{\phi}^2(\mathbf{x}) | i \rangle \ . \tag{4.43}$$

By using the Fourier representation of the δ-function, we may write the S-matrix element for the transition $i \to f$ as

$$M_{fi} = 2\pi\delta(E_f - E_i - q_0)\langle \varphi_F | \hat{V} | \varphi_i \rangle = \lambda \int d^4x \, e^{iq\cdot x} \langle f | \hat{\phi}(x)^2 | i \rangle \ . \tag{4.44}$$

Putting now (4.41) and the last equation into the golden rule (4.40), and summing over the final states of the bath particles we get

$$\sum_f w_{fi} = \frac{\lambda^2}{(2\pi)^3} \mathrm{Re} \int d^4x \int d^3x' \, e^{iq\cdot x} \langle i | \hat{\phi}(\mathbf{x}')^2 \hat{\phi}(x)^2 | i \rangle \ . \tag{4.45}$$

Finally we average the transition rate over the equilibrium distribution of initial states. We then obtain the transition rate per unit volume and time as

$$\frac{1}{VZ(T)} \sum_{i,f} e^{-\beta E_i} w_{fi} = \frac{\lambda^2}{(2\pi)^3} \tilde{\Pi}^{<}(q) \ , \tag{4.46}$$

which expresses the production rate in terms of the Fourier transformed correlation function

$$\tilde{\Pi}^{<}(q) = \int d^4x \, e^{iq\cdot x} \langle \, \hat{\phi}(0)^2 \hat{\phi}(x)^2 \, \rangle \ . \tag{4.47}$$

We may regard the correlation function (4.47) as the self-energy of the Φ-particle in the heat bath. To make contact with the retarded self energy, we recall the relationship (1.53), which is generally valid for any correlation function:

$$\tilde{\Pi}^{>}(q) = e^{\beta q_0} \tilde{\Pi}^{<}(q) \ . \tag{4.48}$$

Furthermore, we may generally define a spectral density through

$$\rho(k) \equiv \tilde{\Pi}^>(q) - \tilde{\Pi}^<(q) = \left(e^{\beta q_0} - 1\right)\tilde{\Pi}^<(q) \tag{4.49}$$

$$= -2\mathrm{Im}\,\tilde{\Pi}^R(q) \,, \tag{4.50}$$

which implies the relationship

$$\tilde{\Pi}^<(q) = -2n(q_0)\,\mathrm{Im}\,\tilde{\Pi}^R(q) \,. \tag{4.51}$$

This is another instance of the general property that all 2-point equilibrium correlation functions may be expressed in terms of the spectral density.

The total production rate per unit volume and per unit time of Φ-particles with energy q_0 is then

$$\begin{aligned}
\Gamma(q_0) &= \lambda^2 \int \frac{d^3q}{(2\pi)^3 2q_0}\tilde{\Pi}^<(q) \\
&= -\lambda^2 \int \frac{d^3q}{(2\pi)^3}n(q_0)\frac{\mathrm{Im}\,\tilde{\Pi}^R(q)}{q_0} \,.
\end{aligned} \tag{4.52}$$

Damping rates, production rates and transport coefficients of thermal system are similarly determined by the imaginary part of the corresponding retarded self energy.

Problem 4.3

Verify that the combination of the imaginary part of the retarded self energy with q_0 in the last factor of (4.52) ensures that this factor is even in q_0.

Chapter 5

Real-Time Formalism

Retarded correlation functions can be obtained by a procedure of analytic continuation of Euclidean (ITFT) Green functions. However, except for the simplest cases, this procedure is usually less than straightforward. Analytic continuation can be avoided entirely by calculating Green functions in the real-time formulation of thermal field theory (RTFT). In this formulation the time-path includes the real-time axis. The original version of RTFT is due to Schwinger (1961) and Keldysh (1963) who considered a closed time path running up and down the real time-axis. The forward and return pieces give rise to a doubling of the degrees of freedom and to a 2×2 matrix structure of propagators and self-energies.

By shifting the back contour down in the complex plane one can construct a whole family of equivalent formulations, which only differ in the explicit form of the propagators. It can be argued on the basis of the analytic structure of the Green functions that the physics is independent of the choice of the back contour. Although there has be some confusion in the past, it is now generally accepted that ITFT and RTFT, in any version, give exactly the same results in perturbation theory. This has now been checked in a great many explicit calculations. Therefore, the choice for ITFT or RTFT very much depends on the problem at hand. For example if one is interested in the classical limit of TFT, the real-time formulation is the most appropriate.

5.1 Generating functional

In the preceding chapter we have seen that the calculation of real-time equilibrium correlation functions necessarily involves a time contour C running up and down the real axis; see Fig 4.1. Let us consider a general thermal Green function

$$
\begin{aligned}
G^{(N)}(x_1, \ldots, x_N) &= \langle\, T_C \hat{\phi}(x_1) \ldots \hat{\phi}(x_N) \,\rangle \\
&= \frac{1}{Z} \int \mathcal{D}\varphi(\mathbf{x}) \langle \varphi(\mathbf{x}); t_i |\, e^{-\beta \hat{H}} T_C \hat{\phi}(x_1) \ldots \hat{\phi}(x_N) | \varphi(\mathbf{x}); t_i \rangle \,, \quad (5.1)
\end{aligned}
$$

defined as the statistical average of a product of Heisenberg fields ordered along the contour C. The ordering instruction T_C orders the operators along the given path C,

that is, it prescribes that the operators it is applied to be arranged in the order in which their time-arguments lie along the oriented contour C; those nearest to the beginning at the right and those nearest to the end to the left. If the contour is parametrically given as a function $t = z(\tau)$ with τ monotonically increasing, the T_C operation is nothing but standard time ordering with respect to τ.

Contour ordering can be formalized by introducing a contour step function and a contour Dirac delta-function according to:

$$\theta_C(t - t') = \theta(\tau - \tau') \,, \tag{5.2}$$

$$\delta_C(t - t') = \left(\frac{\partial z}{\partial \tau} \right)^{-1} \delta(\tau - \tau') \,. \tag{5.3}$$

With these definitions we may write, for example,

$$T_C \hat{\phi}(x) \hat{\phi}(x') = \theta_C(t - t') \hat{\phi}(x) \hat{\phi}(x') + \theta_C(t' - t) \hat{\phi}(x') \hat{\phi}(x) \,. \tag{5.4}$$

Functional differentiation can also be extended in a straightforward manner,

$$\frac{\delta j(x')}{\delta j(x)} = \delta_C(t - t') \delta(\mathbf{x} - \mathbf{x}'), \tag{5.5}$$

for c-number functions $j(x)$ living on the contour.

In the chapter on path integrals we have seen that the FMS path-integral formula may be applied for any contour that goes parallel to the real axis or downward. The derivation proceeds by dividing up the contour in infinitesimal pieces and inserting at each intermediate point a complete set of coherent states. Hence, the contour ordered Green functions may be obtained from the generating functional

$$Z[j] = \int \mathcal{D}\phi \exp i \int_C d^4x [\mathcal{L}(x) + j(x)\phi(x)] \,, \tag{5.6}$$

where $\mathcal{L}(x)$ is the Lagrangian of the system. Because the trace in the thermal average demands that the fields at the begin and end points are the same, and the action of the canonical density operator can be represented as an imaginary time shift, the path integration is over all c-number fields $\phi(x)$ which satisfy the periodic boundary condition

$$\phi(t_i - i\beta, \mathbf{x}) = \phi(t_i, \mathbf{x}) \,, \tag{5.7}$$

where t_i is some chosen initial time. This imposes on the contour C, which defines the action

$$S = \int_C d^4x \, \mathcal{L}(x) \tag{5.8}$$

in the exponent of the path integral (5.6), the condition that it starts at $z = t_i$ and ends at $z = t_i - i\beta$, subject to the monotony condition mentioned earlier.

Contour-ordered thermal Green functions are obtained by functional differentiation with respect to the source in the usual manner. In particular, connected Green functions are obtained from the logarithm of the generating functional

$$G^{(N)}(x_1, x_2, \ldots, x_N) = \left. \frac{\delta^N \log Z[j]}{i\delta j(x_1) \ldots i\delta j(x_N)} \right|_{j=0} \quad . \tag{5.9}$$

As already explained, it is preferable to work with connected Green functions; not only are they diagrammatically simpler objects, but also they have the cluster property, that is, connected Green functions fall off to zero when the distance between the field points increases.

5.2 Perturbation expansion

Formally the path-integral representation for the generating functional (5.6) very much looks like the generating functional for the vacuum Green functions. This allows us to derive a perturbative expansion by the standard reasoning. We split the Lagrangian into a quadratic part and a remainder

$$\mathcal{L}(x) = \mathcal{L}_0(x) + \mathcal{L}_I(x) \quad . \tag{5.10}$$

The quadratic part is given by

$$\mathcal{L}_0(x) = \tfrac{1}{2}\phi(x)\Lambda(i\partial)\phi(x) \quad , \tag{5.11}$$

where $\Lambda(i\partial)$ is a differential operator of finite order. The corresponding contour propagator is defined by the equation

$$\Lambda(i\partial)D_C(x - x') = \delta_C(t - t')\delta(\mathbf{x} - \mathbf{x}') \quad , \tag{5.12}$$

where δ_C is the delta-function on the contour. A unique solution is obtained by specifying the proper boundary condition. For thermal field theory the boundary condition is

$$D_C(t_i - i\beta - t', \mathbf{x}) = D_C(t_i - t', \mathbf{x}) \quad , \tag{5.13}$$

which is just the KMS-condition.

The free generating functional

$$Z_0[j] = \int \mathcal{D}\phi \exp i \int_C d^4x \left[\tfrac{1}{2}\phi(x)\Lambda(i\partial)\phi(x) + j(x)\phi(x)\right] \tag{5.14}$$

may now be calculated in the usual manner by shifting the field according to

$$\phi(x) \to \phi(x) - \int_C d^4x' \, D_C(x - x')j(x') \quad . \tag{5.15}$$

The boundary condition (5.13) guarantees that the shift preserves the periodicity (5.7) of the fields. With the help of Eq. (5.12) we obtain the result

$$Z_0[j] = Z_0[0] \exp -\frac{i}{2} \int_C d^4x d^4x' \, j(x) D_C(x - x') j(x') \quad , \tag{5.16}$$

which is Wick's theorem is this context.

Having established the explicit form of the free generating functional, we now write the full generating functional

$$Z[j] = \int \mathcal{D} e^{iS_0} e^{i(S_I + j \cdot \phi)} \tag{5.17}$$

with $j \cdot \phi = \int_C d^4x \, j(x)\phi(x)$. The Feynman rules follow by associating interaction vertices with S_I and propagator lines with D_C. By expanding out the last exponential in Eq. (5.17), all diagrams of the theory are generated.

In comparison to the vacuum theory there are two new features:
- the time integrations are to be performed along a contour C from an arbitrary time t_i down to $t_i - i\beta$,

- the propagator depends on the contour.

A specific choice of the contour C will lead to a specific formulation of quantum field theory and a corresponding specific set of Feynman rules. It is possible to extend the above considerations to fermionic field theories, and in fact to any theory that can be represented in a path-integral form.

5.3   Contour propagator

As an illustration we calculate the contour propagator for a system of Klein-Gordon particles. One may try to solve (5.12) directly. However, on account of Eq. (5.9) we may identify the propagator obtained from (5.16) as the two-point thermal Green function associated with the free Lagrangian (5.11), apart from a conventional factor i,

$$iD_C(x - x') = \langle T_C \hat{\phi}(x)\hat{\phi}(x')\rangle_0 \quad . \tag{5.18}$$

Comparing now with the thermal propagator (1.58) derived in chapter 1 for time-ordered fields, one easily writes down the solution of the inhomogeneous KG equation (5.12):

$$iD_C(x - x') = \int \frac{d^4k}{(2\pi)^4} \rho_0(k) e^{-ik \cdot (x-x')} [\theta_C(t - t') + \eta n(\omega)] \, , \tag{5.19}$$

where

$$n(\omega) = \frac{1}{e^\omega - \eta} \tag{5.20}$$

is the equilibrium distribution function for both statistics with $\omega \equiv \beta(k_0 - \mu)$ and $\eta = 1$ for bosons and $\eta = -1$ for fermions. The contour step function is zero for times t earlier

than t'. The spectral density already has been defined in (1.49) for scalar particles of mass m and we give the formula once more

$$\rho_0(k) = 2\pi \text{sign}(k_0)\delta(k^2 - m^2) \tag{5.21}$$

$$= -i\tilde{D}^R(k) + i\tilde{D}^A(k) , \tag{5.22}$$

where the propagators in the last line are the free advanced and retarded propagators (1.75) already defined in chapter 1.

This spectral representation of the contour-ordered propagator was first derived by Mills (1969) for nonrelativistic particles. In the case of ordinary time-ordering it reduces to the result of Dolan and Jackiw (1974).

One may verify that the contour propagator indeed satisfies the boundary condition Eq. (5.13). In other words the propagator is periodic in imaginary time. This is the KMS-condition which any two-point equilibrium correlation function should satisfy. It implies that the spectral form (5.19) also holds true for interacting fields, albeit with a different spectral density, because the KMS-condition is equally valid in that case. Similar expressions may be obtained for fermionic and gauge fields.

The spectral representation of the contour propagator is useful for the formal development of real-time thermal field theory, because it is valid for any contour that starts at t_i and goes down to $t_i - i\beta$. For practical calculations, however, we have to be specific. In thermal field theory, historically the standard contour has been the most obvious one, namely a straight line down the imaginary time axis. This leads to the Matsubara formalism as discussed in chapter 2.

We now consider the contour C depicted in Fig 4.1, which allows a direct evaluation of real-time Green functions since the contour contains the real axis. Three segments of the contour $C = C_1 \cup C_2 \cup C_3$ may be distinguished: the segment C_1 that covers the real axis, the return-path C_2 and the Euclidean segment $C_3 = C_E$. It is convenient to define a matrix propagator with components

$$D^{rs}(t - t') = D_C(t_r - t'_s) \quad , \tag{5.23}$$

with t_r on C_r and t'_s on C_s, $r, s = 1, 2, 3$. The times $t = t_2, t_3$ on C_2, C_3 are always later than $t_1 = t$ on C_1. In principle, the two time variables in contour propagator $D_C(x - x')$ can be located on any of the three segments and the Feynman rules would involve a 3×3 matrix of propagators. However, if one takes the limit $t_i, t_f \to \pm\infty$ in an appropriate manner it can be shown that the Euclidean segment decouples from the vertical segments; see the section below. The contour C then reduces to the closed time path $C_1 \cup C_2$, first considered by Schwinger (1961) and Keldysh (1963).

Taking as a fact that in RTFT only the two real-time segments C_1 from $-\infty$ to $+\infty$ and C_2 from $+\infty$ to $-\infty$ need be considered, we may now determine the relevant propagators. In fact these can be extracted immediately from the general expression for the contour propagator (5.19). After a Fourier transformation we get:

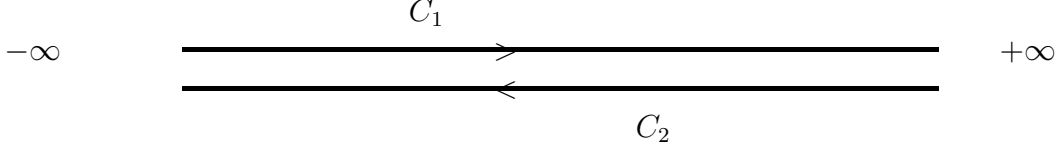$$i\tilde{D}^{11}(k) = \frac{i}{k^2 - m^2 + ik_0\epsilon} + \eta\rho_0(k)n(\omega) , \tag{5.24}$$

$-\infty$ $+\infty$

C_2

Figure 5.1: Closed time path (Keldysh) contour

$$i\tilde{D}^{22}(k) = \frac{-i}{k^2 - m^2 - ik_0\epsilon} + \eta\rho_0(k)n(\omega) , \tag{5.25}$$

$$i\tilde{D}^{12}(k) = \eta\rho_0(k)n(\omega) , \tag{5.26}$$

$$i\tilde{D}^{21}(k) = \rho_0(k)e^\omega n(\omega) . \tag{5.27}$$

The non-thermal terms here are the retarded and advanced propagators (1.75).

Problem 5.1

a. *Show that in momentum space the matrix propagator can be expressed in terms of the Feynman propagator:*

$$i\tilde{D}^{11}(k) = \frac{i}{k^2 - m^2 + i\epsilon} + \eta 2\pi\delta(k^2 - m^2)n(|\omega|) , \tag{5.28}$$

$$i\tilde{D}^{22}(k) = \frac{-i}{k^2 - m^2 - i\epsilon} + \eta 2\pi\delta(k^2 - m^2)n(|\omega|) , \tag{5.29}$$

$$i\tilde{D}^{12}(k) = \eta 2\pi\mathrm{signk}_0\delta(k^2 - m^2)n(\omega) , \tag{5.30}$$

$$i\tilde{D}^{21}(k) = 2\pi\mathrm{signk}_0\delta(k^2 - m^2)e^\omega n(\omega) , \tag{5.31}$$

where $n(|\omega|)$ must be understood as:

$$n(|\omega|) = \frac{\theta(\omega)}{e^\omega - \eta} + \frac{\theta(-\omega)}{e^{-\omega} - \eta} ; . \tag{5.32}$$

Hint: use the relationship

$$\tilde{D}^R(k) = \tilde{D}_{11}(k) - \tilde{D}_{12}(k) = \theta(k_0)\tilde{D}_F(k) - \theta(-k_0)\tilde{D}_F^*(k) . \tag{5.33}$$

b. *In this "Feynman"-representation the vacuum part is cleanly separated from the thermal part. Show that in the vacuum limit $T \to 0$ the distribution function $n(|\omega|)$ vanishes and that the matrix propagator reduces to a trivial diagonal matrix.*

c. *Consider separately the case of a degenerate Fermi system at zero temperature.*

The thermal and vacuum parts of the contour propagator may be separated in an elegant manner by introducing a transformation matrix $M(k)$ such that the matrix propagator is factorized:

$$\tilde{\mathbf{D}}(k) = \mathbf{M}(k)\begin{pmatrix} 0 & \tilde{D}^R(k) \\ \tilde{D}^A(k) & 0 \end{pmatrix}\mathbf{M}^T(-k) . \tag{5.34}$$

Using (5.22), one may verify that the appropriate thermal matrix is

$$\mathbf{M}(k) = \begin{pmatrix} -\eta n(\omega) & 1 \\ n(\omega)\, e^{\omega} & 1 \end{pmatrix} . \tag{5.35}$$

This construction is such that it is invariant under the combined operations of transposition and inversion of k, as is the matrix propagator itself: $\tilde{D}^{rs}(k) = \tilde{D}^{sr}(-k)$. Because in diagrams propagators are always connected to vertices, the factorized form (5.34) could be used to absorb the thermal part entirely in the vertices and work with Feynman rules containing only vacuum propagators.

5.4   Feynman rules

In the preceding section we have already asserted that we only need the two contour pieces C_1, C_2 into account. Formally we define two-component vectors $\phi = (\phi_1, \phi_2)$ and $j = (j_1, j_2)$, where we separated explicitly the two segments of the contour C: $\phi_r(x) = \phi(t_r, \mathbf{x})$, $j_r(x) = (-1)^{r-1} j(t_r, \mathbf{x})$, $t_r \in C_r$, $r = 1, 2$. The two types of fields and sources are to be regarded as independent. The path integral representation (5.17) may then be written in the equivalent form:

$$Z[j_1, j_2] = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \, \exp i \int_{-\infty}^{\infty} d^4 x \left[\phi_r (D^{-1})^{rs} \phi_s + \mathcal{L}_I(\phi_1) - \mathcal{L}_I(\phi_2) + j_r \phi_r \right] . \tag{5.36}$$

The time integral runs from $t = -\infty$ to $t = \infty$ and the minus sign occurs because of the change of time direction on the backward contour. One might have expected also minus signs in the quadratic part and source term. However, in applying (5.36) it is understood that

$$\frac{\delta j_r(t)}{\delta j_s(t')} = \delta_{rs}\delta(t - t'), \quad r, s, = 1, 2 , \tag{5.37}$$

in contrast to what is prescribed by rule (5.5) which implies a minus sign on C_2 due to the negative orientation. This minus sign is absorbed in the definition of the source $j_2(x)$. From now on we adopt the new sign covention. In passing we remark that boundary conditions are unimportant because they are only needed for the determination of the propagators which we already know.

The generating functional (5.36) makes it evident that an effective doubling of the degrees of freedom is necessary in order to be able to calculate real-time Green functions. From the generating functional, real-time Green functions are obtained by functional differentiation:

$$G^{(N)}(x_1, x_2, \ldots, x_N) = \left. \frac{\delta^N \log Z[j]}{i\delta j_1(x_1) \ldots i\delta j_1(x_N)} \right|_{j_1 = j_2 = 0} , \tag{5.38}$$

with respect to the c-numbers sources $j_1(x)$. This implies that only type-1 fields appear on external lines. The type-2 fields may be interpreted as some sort of "ghost" field, since

they only appear on internal lines. This doubling is absent in the Matsubara formalism and one might be inclined to consider it as a mathematical artefact. However, in the axiomatic formulation of quantum statistical mechanics a similar doubling occurs. This indicates that a two-component formulation is the prerequisite for a consistent Minkowski-space thermal field theory.

We are now in the position to state the Feynman rules. In the expansion of (5.36) two types of vertices occur, one type describing the interactions of the original real-time field ϕ_1 and the other the interactions of the thermal ghost field ϕ_2. The Feynman rules for these two types of vertices differ only by a minus sign. There is no direct coupling between the two fields, but they can propagate into each other because of the non-diagonal elements of the matrix propagator. External lines of physical Green functions are always of type-1. Therefore, to find a particular Green function $\tilde{G}^{1\cdots 1}(k_1, \ldots, k_N)$ in momentum space, we must draw all diagrams with N external lines of type-1, and an arbitrary number of vertices of type $r = 1, 2$. These two types of vertices are connected to each other by (directed) propagator lines representing $\tilde{D}^{rs}(k)$.

We now summarize the rules for a scalar $\lambda\phi^4$-theory:

 (i) Draw diagrams and determine symmetry factors as in vacuum field theory.

 (ii) Assign a propagator $i\tilde{D}^{rs}(k) = \langle\phi_r\phi_s\rangle_0 = r \;\underline{\hspace{2cm}}\; s$ to each line connecting vertices.

 (iii) Assign a factor $-i\lambda$ to each type-1 vertex and a factor $i\lambda$ to each type-2 vertex.

 (iv) Conserve energy and momentum at each vertex according to the usual prescription: $(2\pi)^4\delta(\sum_i k_i)$; global energy and momentum conservation may be separated off by excluding one arbitrarily chosen vertex.

 (v) Integrate over all internal momenta and energies according to $\int d^4k/(2\pi)^4$; sum over all values of the internal indices $r = 1, 2$.

Compared to the Feynman rules of the vacuum theory, the difference is the matrix propagator and the occurrence of two type of vertices.

As in the vacuum theory disconnected blobs without external legs may be ignored. In fact they are exactly equal to unity, that is, the sum of all bubble diagrams, with vertices occurring once as type-1 and once as type-2 (and hence with a minus sign), exactly vanishes. This may be expressed as

$$\log Z[j_1 = 0, j_2 = 0] = 0 \;. \tag{5.39}$$

This theorem is rather obvious from the defining equation (5.36). It was first discovered in the context of the Keldyhs formulation of TFT.

5.5 Keldysh formulation

Calculations of real-time Green functions become rather cumbersome at higher order in the coupling constants. Moreover, the quantities of the most interest are the retarded

en advanced Green functions, and not the time-ordered ones. Therefore we discuss in this section a set of rules which is more economical and is aimed at obtaining retarded and advanced Green functions. This is a variation on the real-time formulation of finite temperature field theory, based on the observation that the two-point Green functions satisfy the identity

$$G^{11}(x,y) - G^{12}(x,y) - G^{21}(x,y) + G^{22}(x,y) = 0 , \qquad (5.40)$$

which can be checked directly from the their definitions. Similar relations hold for higher-order Green functions,

$$\sum_{s_1,\dots,s_N} (-1)^{\#\{i|s_i=-1\}} G^{(N)}_{s_1,\dots,s_N}(x_1,\dots,x_N) = 0 . \qquad (5.41)$$

These relations are often referred to as largest-time-equations (LTE), because they follow from the fact that the two segments of the contour C are connected at $t = \infty$.

The Keldysh-formulation makes use of the LTE (5.40) to reduce the number of propagators to three. First we change notation; from now on we denote the two branches of the contour as C_+, C_- and the corresponding fields and sources $\phi_\pm, j_\pm$. Subsequently, we perform a linear transformation of the fields $\phi_{+,-}$ to a "classical" field ϕ_1 and a "quantum" field ϕ_2 according to

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} (\phi_+ + \phi_-)/2 \\ (\phi_+ - \phi_-)/\hbar \end{pmatrix}, \qquad (5.42)$$

such that the (free) matrix propagator takes the form

$$i\hbar \mathbf{D}(x - x') \rightarrow \begin{pmatrix} F_0(x-x') & i\hbar D^R(x-x') \\ i\hbar D^A(x-x') & 0 \end{pmatrix}. \qquad (5.43)$$

Here the free retarded and advanced propagator functions are given in momentum space by the expressions (1.75), whereas the thermal Keldysh propagator in momentum space reads

$$
\begin{aligned}
\tilde{F}_0(k) &= \tfrac{1}{2}\hbar \coth(\hbar\omega/2)\rho_0(k) \\
&= \hbar \sum_{s=\pm} [n(s\omega_k) + \tfrac{1}{2}] \frac{1}{2\omega_k} 2\pi s \delta(k^0 - s\omega_k) .
\end{aligned} \qquad (5.44)
$$

The retarded and thermal two-point functions are related by the KMS condition (fluctuation-dissipation theorem)

$$\tilde{F}_0(k) = i\hbar\, n(\hbar\omega) \left[\tilde{D}^R(k) - \tilde{D}^A(k) \right]. \qquad (5.45)$$

In these formulae we have explicitly displayed $\hbar$ to indicate the difference between the quantum thermal part $\tilde{F}_0$, and the retarded and advanced propagators which are basically classical.
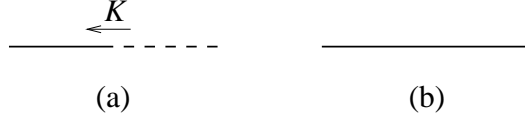
Figure 5.2: Propagators: (a) $\tilde{D}^R(k) = \tilde{D}^A(-k)$, (b) $\tilde{F}_0(k)$.

Figure 5.3: Vertices: (a) $\frac{1}{3!}\lambda\phi_1^3\phi_2$, (b) $\frac{1}{4!}\lambda\phi_1\phi_2^3$.

Feynman rules appear when also the interaction part along the closed time path contour is written in terms of the $\phi_{1,2}$ fields

$$
\begin{aligned}
S_I &= -\int d^4x \left(\frac{1}{4!}\lambda\phi_+^4 - \frac{1}{4!}\lambda\phi_-^4 \right) \\
&= -g^2 \int d^4x \left(4\phi_1^3\phi_2 + \hbar^2\phi_1\phi_2^3 \right) ,
\end{aligned} \tag{5.46}
$$

where we defined the dimensionless coupling constant $g^2 = \lambda\hbar/24$.

The rules are presented pictorially in figs. 5.2 and 5.3. The ϕ_1 field is denoted with a full line and the ϕ_2 field with a dashed line. The contractions that appear are

$$
\begin{aligned}
\langle \phi_1(x)\phi_1(x') \rangle_0 &= F(x - x') , &(5.47) \\
\langle \phi_1(x)\phi_2(x') \rangle_0 &= iD^R(x - x') , &(5.48) \\
\langle \phi_2(x)\phi_1(x') \rangle_0 &= iD^A(x - x') . &(5.49)
\end{aligned}
$$

Since retarded and advanced propagators interpolate between a ϕ_1 and an ϕ_2 field, they are indicated with a dashed-full line. For the retarded and advanced Green functions, it is necessary to specify the direction of the momentum flow through the propagator, and this is indicated with the arrow.

The Feynman rules are the same as before, except:

(i) Thermal correlation functions have N full legs.

(ii) The retarded self energy and the so-called generalized retarded N-point functions have one dashed leg and $N-1$ full legs. These are shown in fig. 5.4. The arrows denote again the momentum flow of the external momenta.
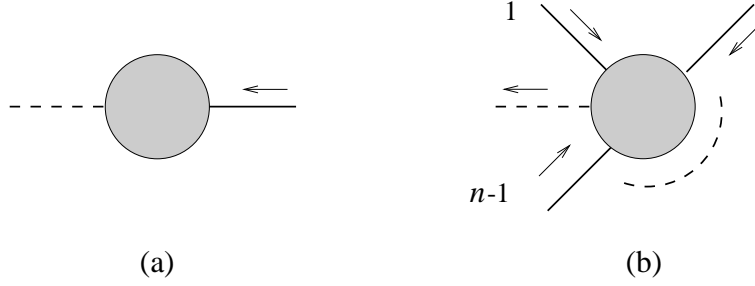
Figure 5.4: (a) Retarded self-energy, (b) generalized retarded N-point functions.

(iii) Diagrams are constructed using vertices (a) and (b).

One may note that vertex (b) can only appear in a diagram with retarded (or advanced) propagators attached to the three dashed legs. After attaching these propagator functions, the resulting outer lines (which either still have to be attached to another vertex or are external lines) are always full lines. However, such a configuration can be constructed as well with vertex (a); this vertex has two full lines where vertex (b) has two dashed legs. By attaching two thermal two-point functions on these legs, the external lines are full as well, and the vertices can be part of a diagram in exactly the same manner, but with two more thermal functions.

As a simple example we compute the retarded one-loop self energy for the three-point coupling

$$
\begin{aligned}
S_I &= -\int d^4x \left(\frac{1}{3!}\phi_+^3 - \frac{1}{3!}\phi_-^3 \right) \\
&= -\hbar \int d^4x \left(\tfrac{1}{2}\phi_1^2\phi_2 + \frac{\hbar^2}{4!}\phi_2^3 \right) ,
\end{aligned}
\tag{5.50}
$$

where we have set the coupling constant equal to unity for simplicity. A retarded diagram begins with a full line and ends with a dashed line. At one-loop we get two diagrams as drawn in the figure. The thermal contribution is

$$
\tilde{\Pi}^R(p) = \int \frac{d^4k}{(2\pi)^4} \, \tilde{F}(k)\tilde{D}^R(p-k) .
\tag{5.51}
$$

Substsituting the representations (1.75) and (5.44) we arrive at:

$$
\tilde{\Pi}^R(p) = \sum_{s,s'=\pm 1} \int \frac{d^3k}{(2\pi)^3} \frac{s}{2\omega_k} \frac{s'}{2\omega_{p-k}} \frac{[n(s\omega_k) + n(s'\omega_{p-k}) + 1]}{p^0 + i\epsilon - s\omega_k - s'\omega_{p-k}} ,
\tag{5.52}
$$

where p_0 is the real external energy. This form looks very much like the photon self energy, we have calculated in chapter 3. The calculation of this quantity with the Keldysh rules, we leave as an exercise.
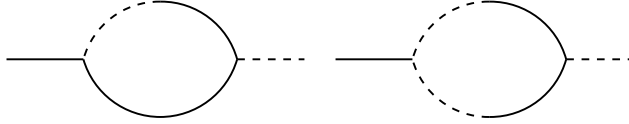
Problem 5.2

Figure 5.5: Thermal and vacuum self-energy for a three-point coupling.

Calculate the retarded photon self energy

$$\tilde{\Pi}^R_{\mu\nu}(k) = \tilde{\Pi}^{11}_{\mu\nu}(k) + \tilde{\Pi}^{12}_{\mu\nu}(k) \tag{5.53}$$

with the Keldysh rules.

5.6 Some matters of principle

Above we have stated without proof that in the calculation of real-time Green functions, the imaginary part of the contour can be ignored when $t_i \to -\infty$. This implies that the generating functional factorizes into a contribution from the two infinite contours $C_1 \cup C_2$ and one from the imaginary part C_E :

$$Z[j] = Z_{12}[j]Z_{\mathrm{E}}[j] \quad . \tag{5.54}$$

The last factor is the Euclidean generating functional discussed in the preceding chapter. However, for the calculation of real-time Green functions with time arguments either on C_1 or C_2, this factor is just a multiplicative constant which entirely drops out.

To see how the factorization comes about, it is useful to consider the perturbative expression for an N-point thermal Green function

$$G(x_1, \ldots, x_N) = \frac{\langle \phi(x_1) \ldots \phi(x_N) \exp i \int_C d^4x \mathcal{L}_I(x) \rangle_0}{\langle \exp i \int_C d^4x \mathcal{L}_I(x) \rangle_0} \quad . \tag{5.55}$$

The contour C includes the imaginary segment C_E. The result is analogous to the well-known GML-formula of zero-temperature field theory. For factorization in the sense as discussed above to occur, it is not sufficient that the contributions from the imaginary segment C_{E} in numerator and denominator cancel, because in the connected diagrams the time integral still runs from t_i to $t_i - i\beta$. However, equilibrium correlation functions can only depend on time differences. Hence any dependence on t_i should drop out and this is exactly what one finds in explicit calculations.

The general argument to prove this goes as follows. Because equilibrium thermal Green functions are analytic in the strip $-\beta < \mathrm{Im}\, t < 0$, we may deform the contour $C_3 = C_E$ to first run back to some earlier time t_i^*, then down an Euclidean path C_3^*, and
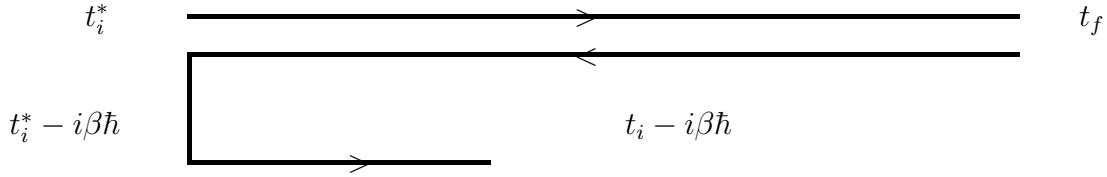
Figure 5.6: Contour used in the proof of the factorization of Green functions

finally parallel to the real time back to $t_i - i\beta$. Because of the periodicity of thermal Green functions, we may then write the internal time-integration of some propagator as

$$\int_0^\beta d\tau\, G^{13}(t_i - i\tau, \mathbf{x}, x') = \int_0^\beta d\tau\, G^{13}(t_i^* - i\tau, \mathbf{x}, x') + i \int_{t_i^*}^{t_i} dt \left[G^{11}(x, x') - G^{12}(x, x') \right] \quad (5.56)$$

for arbitrary t_i^*, which we now can take to $-\infty$. On account of the cluster property (2.79), the first term at the right-hand side may be dropped. Note that for the cluster property to apply, at least one external real time in any connected Green function must be fixed on either C_1 or C_2. The second term extends the time integration to the infinite closed time path $C_1 \cup C_2$, first considered by Schwinger (1961) and Keldysh (1963). The cancelation of the acausal parts of the time contour is a general feature of the time-path method, because equal and opposite contributions will always come from the portions of the integration contour C_{12} where $t' > t$. In practice this means that in the calculation of real-time Green functions, the contours can taken to be inifnite, and all contributions from the Euclidean contour simply may be ignored. This is the infamous decoupling theorem of thermal field theory.

In the standard derivations of the Schwinger-Keldysh formulation, one arrives at this closed time path by imposing an adiabatic condition, which is tantamount to ignoring the contour C_E. However, it may be pointed out that the imaginary contribution is essential in order that the KMS condition be satisfied (except in the case where the fields in Eq. (5.55) carry the same masses as the asymptotic fields in the usual sense of the vacuum field theory). Furthermore, shrinking the contour C to the imaginary segment, formula (5.55) reduces to the standard GML-formula of ITFT.

The point we want to emphasize here is that factorization is a property of the interaction. For short-range interactions it may be expected that the (connected) Green functions relax to zero on the scale of some finite relaxation time and factorization occurs. However, special care should be taken in the case of zero-energy modes. In that case the cluster property may not apply in the usual manner. Still in most models considered in RTFT the factorization seems to be valid, since the results are in complete agreement with ITFT calculations.

The closed path contour is by no means unique. A direct evaluation of real-time Green functions is always possible if the time contour includes the real axis. The back contour is more or less arbitrary. The simplest family of such "real-time" contours is depicted in Fig 5.6. The path goes from t_i to t_f along the real axis, drops vertically from t_f to $t_f - i\sigma\beta\hbar$, returns parallel to the real axis to $t_i - i\sigma\beta\hbar$, and ends at $t_i - i\beta\hbar$. By
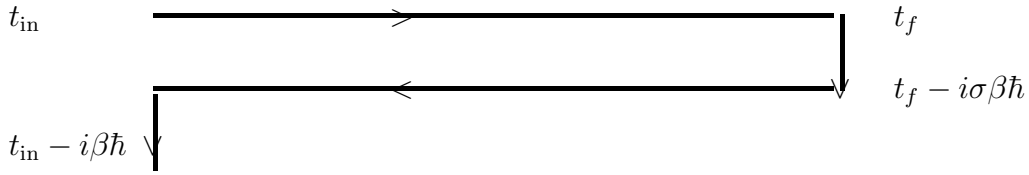
Figure 5.7: Time contours labeled by the parameter $0 \leq \sigma \leq 1$.

varying $0 \leq \sigma \leq 1$ an equivalence class of thermal field theories is generated. The choice $\sigma = 0$ corresponds to the Schwinger-Keldysh formalism, and is in some sense the simplest. Physical results should not depend on the choice of σ, and this can be shown to be the case; Matsumoto, Nakano, Umezawa, Mancini, Marinaro (1983).

There is a final matter to be cleared up. In the real-time formalism one might expect to encounter certain "pathologies". In this context a pathology is a so-called pinch singularity. Singularities of this type occur when an expression like

$$\tilde{\Delta}_F \tilde{\Delta}_F^* = \frac{1}{k^2 - m^2 + i\varepsilon} \frac{1}{k^2 - m^2 - i\varepsilon} \tag{5.57}$$

is integrated over k_0. In the limit $\epsilon \to 0$ the integration is then "pinched" between the poles on each side of the real axis, and the integral cannot be assigned a well-defined meaning. Another example of such an expression without meaning is

$$\frac{\delta(k^2 - m^2)}{k^2 - m^2 + i\varepsilon} = -\tfrac{1}{2}\delta^{(1)}(k^2 - m^2) - i\pi[\delta(k^2 - m^2)]^2 \tag{5.58}$$

in which a product of delta-functions occurs with the same argument. With the real-time propagators (5.31) there certainly seems to be a danger that such terms appear.

These problems are avoided by working with the regularized form of delta-functions and their derivatives:

$$2\pi i \delta^{(n)}(x) = \left(\frac{-1}{x + i\varepsilon}\right)^{n+1} - \left(\frac{-1}{x - i\varepsilon}\right)^{n+1} . \tag{5.59}$$

If ϵ is kept finite, the pinch singularities are regularized in the intermediate stages of a particular calculation. It is then a general rule that potentially dangerous terms will cancel after all relevant diagrams have been taken into account. This property has been shown to be a consequence of the real-time Feynman rules by Niemi and Semenoff (1984), and Matsumoto, Ojima, Umezawa (1984). The full matrix structure of the theory is instrumental in the proof. This explains why in early attempts at a real-time approach, in which one only considered the propagator $\tilde{D}^{11}(k)$, one was faced with inadmissible singularities of the pinch type. We like to add that it is a great advantage of the Keldysh formulation that the cancellations are immediately evident, before any momentum integrals are done.

Chapter 6

High-Temperature Dimensional Reduction

The complicated nature of the dynamics of interacting thermal field theories has motivated the search for a regime where these theories simplify in some sense. Conventional wisdom has it that the infinite-temperature limit, or the critical region close to a phase transition, provides such a regime, in that the four-dimensional field theory reduces to a three-dimensional one. The argument here centers upon the fact that in weakly coupled field theories, the temperature T can be used as expansion parameter to effectively isolate the zero-frequency mode of the theory. This is done by integrating out all non-zero frequency modes which, having large effective masses of order T in their propagators, are not plagued by IR problems. Their effects can be treated as corrections, systematically calculable order by order in perturbation theory, to an effective 3D theory of the remaining zero-frequency mode. To the effective 3D-theory then various methods can be applied: ϵ expansion, exact renormalization group, gap equations, Monte Carlo simulations, etc. Thus, a combination of perturbative and non-perturbative methods is to be used to solve the problem.

Such a dimensional reduction would apply to static correlation functions at small momenta $k \ll T$. This would simplify the description of finite-temperature phase transitions such as, for example, the chiral symmetry restoration transition in QCD, or the deconfinement transition in pure gauge symmetries. Dimensional reduction also plays an important role in the study of the electro weak phase transition.

Dimensional reduction (DR) in thermal field theory has attracted considerable interest the last few years. The original idea of this construction, goes back to papers by Ginsparg (1980), and by Appelquist and Pisarski (1981). DR was further developed by Farakos, Kajantie, Laine, Rummukainen, and Shaposhnikov, in a study of the electroweak transition and applied to hot QCD by Braaten and Nieto (1995). Different aspects of dimensional reduction were studied by Landsman (1989), Jackiw and Templeton (1981) and Jakovac, Kajantie, and Patkos (1994).
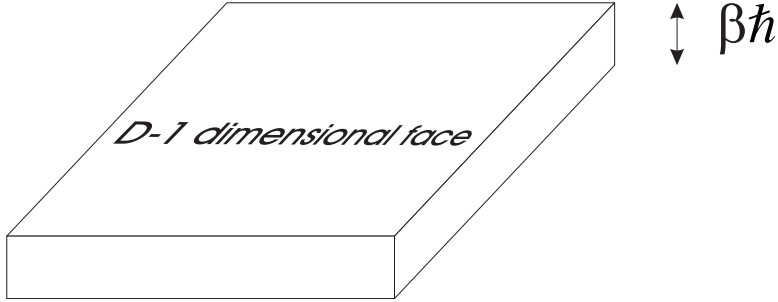
Figure 6.1: Flat box in d-dimensions, of thickness $\hbar\beta$.

6.1 Static dimensional reduction

The traditional description of DR is based on the Euclidean formalism. In this approach the system is living in a box of $d+1$ dimensions. We assume that the extent in d spatial dimensions is much larger than the scale of the physical correlations. The extent of the Euclidean dimension is $\hbar\beta$, with $\beta = 1/T$ the inverse temperature. In this picture dimensional reduction has a simple geometric origin: fluctuations with wavelength larger than the extent in the Euclidean time-direction are "squeezed" in this direction and behave like d-dimensional fluctuations. In other words for these fluctuations the system looks like a "pancake". If such fluctuations determine the physics of a phase transition then this physics is effectively d-dimensional.

There is a close connection between DR and the classical limit. This rests on the observation that the long-wave-length fluctuations at high T are just classical thermal fluctuations. In Euclidean thermal field theory this is easy to see. One typically sums over Matsubara frequencies, for example

$$\frac{T}{\hbar} \sum_n \frac{\omega}{\omega^2 + (2\pi n T/\hbar)^2} = \frac{1}{\beta\hbar\omega} + \frac{2}{\beta\hbar} \sum_{n=1}^{\infty} \frac{\omega}{\omega^2 + (2\pi n T/\hbar)^2} \tag{6.1}$$

$$= \frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} + \ldots . \tag{6.2}$$

The term $1/\beta\hbar\omega$ comes from the zero mode, whereas the first term in the last line is the vacuum contribution. In the limit $\beta\hbar\omega \ll 1$ one may expand

$$\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} = \frac{1}{\beta\hbar\omega} + \frac{1}{12}\beta\hbar\omega_k + \ldots \tag{6.3}$$

The series has a radius of convergence of $\beta\hbar\omega = 2\pi$, and for small expansion parameter $\beta\hbar\omega \ll 1$ it is approximated extremely well by its leading term. Hence, since the first term in (6.1) is equal to the $n = 0$ term in the Matsubara sum, DR amounts to taking the classical limit $n(\omega) \to 1/\beta\hbar\omega$, dropping all sub-leading terms in the Bose-Einstein distribution function $n(\omega)$. Thus, modes with $\hbar\omega \ll T$ are described by a classical statistical theory. For the photon gas, for example, this is the Rayleigh-Jeans region of frequencies. The classical behavior of a field represented by these modes is what the

dimensionally reduced theory describes. In quantum field theory this regime applies either when $T \to \infty$ or at a second order phase transition as $T \to T_c$ and $\omega \to 0$.

In classical statistical mechanics we would make the identification: $S^E[\phi]/\hbar = \beta E[\phi]$, where $E[\phi]$ is the energy functional of the classical field. This we recover indeed in the classical limit $\hbar\beta \to 0$

$$\frac{1}{\hbar}S^E[\phi] = -\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \int d^3x \ \mathcal{L}(-i\tau, \mathbf{x})) \to \beta \int d^3x \mathcal{H} \ , \tag{6.4}$$

with $\mathcal{H} = (\partial_t \phi)^2 - \mathcal{L}$ the Hamiltonian density of the system.

Dimensional reduction rest on the fact that there is an energy gap in the energy spectrum. This comes out even clearer when we consider the expansion of the Euclidean field in Matsubara modes

$$\phi(\tau, \mathbf{x}) = \sqrt{T} \sum_n e^{-i\omega_n \tau} \phi_n(\mathbf{x}) \ . \tag{6.5}$$

For the spatial correlator we get

$$\langle \phi_n(\mathbf{x})\phi_{-n}(0)\rangle_0 = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + (m/\hbar)^2 + \omega_n^2} \ . \tag{6.6}$$

The Matsubara frequencies act like temperature-dependent mass terms. Taking the vacuum mass $m = 0$, we may define a hierarchy of decreasing correlation lengths

$$\xi_n^2(T) = \frac{\hbar}{2\pi nT} \ . \tag{6.7}$$

We then see that the contribution to the correlator from the exchange of a Fourier mode with frequency ω_n falls off at large $r = |\mathbf{x}|$ as

$$\langle \phi_n(\mathbf{x})\phi_{-n}(0)\rangle_0 = \frac{e^{-r/\xi_n(T)}}{4\pi r} \ . \tag{6.8}$$

Thus the only modes whose contributions do not fall off exponentially at distances $r \gg \hbar\beta$ are the $n = 0$ modes of the bosons.

The sum $n \neq 0$ in (6.1) can be interpreted as a sum over particles with masses $2\pi nT$ in the perturbation theory of a $3D$-dimensional field theory at temperature zero (Ginsparg 1980, Jourjine 1984). This suggests the strategy of integrating out the fermionic modes and the nonzero modes of the bosons to get an effective theory for the bosonic zero modes. This process is called "dimensional reduction". It results in a 3D Euclidean field theory with bosonic fields only, which reproduces the static correlators of the original theory at distances $r \gg \hbar/T$.

Problem 6.1

Calculate the 3D-correlator

$$\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2 + \xi^{-2}} = \frac{e^{-r/\xi}}{4\pi r} \tag{6.9}$$

Hint: use a contour of integration that encloses the pole $z = i\xi^{-1}$.

In the vacuum theory this decoupling of "heavy" and "light" particles at all orders in perturbation theory is the content of the Appelquist-Carrazzone decoupling theorem (1975). However, the 3D effective thermal field theory typically generates a dynamical mass gT and the correction terms do not vanish in the high-T limit. As we will see this restricts the validity of DR to a certain order in g.

Another obvious difficulty is of course the ultra-violet catastrophe: the total energy diverges because of the contribution of the high-frequency modes. This means that the dimensionally reduced theory can only be an effective theory for the "soft" modes with $\hbar\omega \ll T$. The UV-divergences have to be regularized by quantum theory. However, one expects the "hard" modes to behave perturbatively since the dominant IR behavior is carried by the soft modes. It should be possible at least in principle to integrate out the hard modes and to obtain an effective theory for the soft modes. The effect of the "hard" modes is then to renormalize the couplings of this effective theory.

The point here is that renormalization schemes in this context are not just about getting rid of UV divergences, but about defining a set of parameters with which to describe the theory and to be able to calculate quantities of physical interest. In the case of DR one should consider a T-dependent renormalization scheme.

6.2 Effective theory

The derivation of effective theories is one of the fundamental problems in statistical mechanics. At the basic microscopic level many-particle systems have nearly an infinite number of degrees of freedom. In particular experimental situations one usually is interested only in the behavior of a rather restricted set of observables. To mention a few examples:

(i) Hydrodynamics; the relevant variables are the conserved currents $J^\mu(x)$ and conserved energy momentum tensor $T^{\mu\nu}(x)$;

(ii) Phase transitions; the relevant variable is the order parameter such as there is

- the magnetization for ferromagnetic systems

$$\mathbf{m}(\mathbf{x}) = \langle \sum_i \mathbf{s}_i \delta(\mathbf{x}_i - \mathbf{x}) \rangle \,, \tag{6.10}$$

- the condensate amplitude for superfluid systems

$$\varphi(\mathbf{x}) = \langle \phi(\mathbf{x}) \rangle \,, \tag{6.11}$$

- the electron pair amplitude for superconducting systems

$$\Delta(\mathbf{x}) = \langle \psi_\downarrow(\mathbf{x})\psi_\uparrow(\mathbf{x}) \rangle \ . \tag{6.12}$$

In dimensional reduction the relevant variable is the zero mode $\Phi(\mathbf{x}) = \phi_0(\mathbf{x})$ of the Euclidean field expanded in Matsubara modes

$$\phi(\tau,\mathbf{x}) = \sqrt{T}\Phi(\mathbf{x}) + \sqrt{T}\sum_{n\neq0} e^{-i\omega_n\tau}\phi_n(\mathbf{x}) \ . \tag{6.13}$$

The reduction of the description from the microscopic degrees of freedom to a smaller set of variables may be pictured as a projection of the full microscopic theory onto a subspace of relevant variables. The projector in DR is

$$\Phi(\mathbf{x}) = \mathcal{P}\phi(\tau,\mathbf{x}) = \frac{\sqrt{T}}{\hbar}\int_0^{\beta\hbar} d\tau\ \phi(\tau,\mathbf{x}) \ . \tag{6.14}$$

Factors $\hbar$ are written out explicitly to facilitate the discussion of the classical limit. In units with $k_B = 1, c = 1$ the mode fields $\phi_n(\mathbf{x})$ have the dimension $\sqrt{1/L}$.

When a relevant variable or order parameter has been identified, the effective theory may be derived by the following general reasoning (Fukuda Kyriakopoulos, 1975; O'Raifeartaigh, Wipf, Yoneyama, 1986). Let us consider in particular the generating functional of the scalar theory

$$Z[j] = \int \mathcal{D}\phi(\tau,\mathbf{x})e^{-S^E[\phi]+j\phi} \ , \tag{6.15}$$

where $j(\mathbf{x})$ is an static external source that couples to the zero-mode of the field

$$j\phi \ = \ \frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3x\, j(\mathbf{x})\phi(\tau,\mathbf{x}) \equiv J\cdot\Phi \ , \tag{6.16}$$

$$J\cdot\Phi \ \equiv \ \int d^3x J(\mathbf{x})\Phi(\mathbf{x}) \ , \tag{6.17}$$

with $J(\mathbf{x}) = \sqrt{\beta}j(\mathbf{x})$. From this generating functional we get the static correlation functions by functional differentiation

$$\langle\Phi(\mathbf{x}_1)\ldots\Phi(\mathbf{x}_N)\rangle = \left.\frac{\delta^N \log Z[J]}{\delta J(\mathbf{x}_1)\ldots\delta J(\mathbf{x}_N)}\right|_{J=0} \ ;. \tag{6.18}$$

We now define the so-called constrained effective action (CEA) as

$$e^{-U[\Phi]} = \mathcal{N}\int \mathcal{D}\phi(\tau,\mathbf{x})\ \delta(\Phi - \mathcal{P}\phi)\ e^{-S^E[\phi]} \ . \tag{6.19}$$

It may be regarded as a probability density for the system to be in a state described by the static field $\Phi(\mathbf{x})$, as it is the sum of the probabilities of the system being in each of the microscopic states that are allowed by the constraint $\Phi - \mathcal{P}\phi = 0$; $\mathcal{N}$ is an arbitrary

normalization factor. Directly from the definition of the CEA we may write the generating functional as a functional over the static field

$$Z[J] = \mathcal{N} \int \mathcal{D}\Phi(\mathbf{x}) \, e^{-U[\Phi]+J\cdot\Phi} \tag{6.20}$$

from which the static correlation functions are calculated as averages

$$\langle \Phi(\mathbf{x}_1)\ldots\Phi(\mathbf{x}_N) \rangle = \frac{1}{Z} \int \mathcal{D}\Phi(\mathbf{x})\Phi(\mathbf{x}_1)\ldots\Phi(\mathbf{x}_N) \, e^{-U[\Phi]} \tag{6.21}$$

with probability density $\sim \exp -U[\Phi]$.

We now rewrite the delta-function in (6.19) as an integral

$$\delta(\Phi - \mathcal{P}\phi) \sim \int_{-\infty}^{\infty} dJ \exp -J \cdot (\Phi - \mathcal{P}\phi) \,. \tag{6.22}$$

This is a generalized Fourier transform with a contour of integration in the complex plane which is irrelevant for the algebraic considerations which follow. Substituting into the definition of CEA (6.19), and taking the definition of the generating functional into account, we get

$$e^{-U[\Phi]} = \mathcal{N} \int \mathcal{D}J \, e^{-J\cdot\Phi} \, e^{\log Z[J]} \,. \tag{6.23}$$

where $Z[J]$ is the generating functional (6.15). Thus, the CEA is a Fourier transform of the ordinary generating functional. In the thermodynamic limit, the integral may be calculated by expanding around the stationary point

$$\frac{\delta \log Z[J]}{\delta J(\mathbf{x})} = \langle \mathcal{P}\phi(\mathbf{x}) \rangle_J = \Phi(\mathbf{x}) \,. \tag{6.24}$$

Let us call the solution of this mean-field equation $J_0(\mathbf{x})$. Then in the infinite volume limit the result is dominated by the mean-field-approximation

$$\lim_{V\to\infty} U[\Phi] = J_0 \cdot \Phi - \log Z[J_0] \equiv \Gamma[\Phi] \,. \tag{6.25}$$

The right-hand-side may be recognized as the Legendre transform by which one defines the *effective action* in field theory. Hence, the constrained effective action and the effective action are one and the same in the mean-field approximation.

As is discussed in many textbooks, the functional expansion coefficients of the effective action are the vertex functions

$$\Gamma[\Phi] = \sum_N \frac{1}{N!} \int d^3x_1 \ldots d^3x_N \, \Gamma^{(N)}(\mathbf{x}_1,\ldots,\mathbf{x}_N)\Phi(\mathbf{x}_1)\ldots\Phi(\mathbf{x}_N) \,. \tag{6.26}$$

Diagrammatically, the vertex functions are represented by one-particle irreducible (1PI) diagrams, that is, diagrams that cannot become disconnected by cutting any single line. The 1PI-diagrams are amputated meaning that there are no propagators on the external

lines. An elegant proof has been given by Brandenberger (1985) with the help of the identity

$$\frac{1}{a}\log Z[J] = \log \int \mathcal{D}\Phi e^{-\frac{1}{a}(\Gamma[\Phi]-J\cdot\Phi)} \tag{6.27}$$

which holds true to lowest order in the parameter a. Indeed, for small a the extremum of the exponential at the value $\Phi_0(\mathbf{x})$, which is the solution of the equation

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(\mathbf{x})} = J(\mathbf{x}) , \tag{6.28}$$

becomes very sharp and we may apply the mean-field (stationary-phase) approximation:

$$\frac{1}{a}\log Z[J] \simeq \frac{1}{a}(J_0 \cdot \Phi_0 - \Gamma[\Phi_0]) . \tag{6.29}$$

This is precisely the inverse of the Legendre transform in (6.25).

The proof now proceeds by noting that the effective action $\Gamma[\Phi]$ may be regarded as the action of a new effective theory with vertices $\Gamma^{(N)}, N \geq 3$ and inverse propagator

$$\left[G^{(2)}(\mathbf{x},\mathbf{x}')\right]^{-1} = \Gamma^{(2)}(\mathbf{x},\mathbf{x}') = \Delta^{-1}(\mathbf{x},\mathbf{x}') + \Pi(\mathbf{x},\mathbf{x}') , \tag{6.30}$$

where Δ is the free propagator of the theory and Π the self energy defined by $\Gamma^{(2)}$. For this new theory we can set up a perturbation theory in the usual manner, except that now

- every vertex $\Gamma^{(N)}$ is multiplied by the factor $1/a$, and

- every propagator $G^{(2)}$ is multiplied by a.

Therefore, to any given order in perturbation theory every amputated diagram is multiplied with the factor a^{I-V}, with $I = \#$ of internal lines, and $V = \#$ of vertices. With the help of the Euler formula $L - I + V = 1$, the number of vertices and internal lines can be related to the number of loops L, which may be defined as the smallest number of lines in a diagram that must be removed to make it a tree-graph, that is, a graph without loops. We thus find that any diagram is multiplied by a^{L-1}, which means that the expansion in a is also an expansion in the number of loops. In particular $L = 0$ corresponds to the mean-field approximation.

Now, to lowest order in a, the right-hand-side of (6.27) consists of a sum of tree diagrams in terms of the full vertices $\Gamma^{(N)}, N \leq 3$ and propagator $G^{(2)}$. According to the theorem, this sum is equal to the sum of all connected diagrams in terms of the free propagator and bare vertices. Hence, the vertex functions may be identified with the 1PI-diagrams of the original theory.

Problem 6.2

Show by functional differentiation of (6.28) that the full propagator satisfies the reciprocity relation

$$\Gamma^{(N)}(1,2) = \frac{\delta^2 \Gamma[\Phi]}{\delta\Phi(1)\,\delta\Phi(2)} = \left[\frac{\delta^2 \log Z[J]}{\delta J(1)\,\delta J(2)}\right]^{-1} = \left[G^{(2)}(1,2)\right]^{-1} \tag{6.31}$$

and thus is the inverse of $\Gamma^{(2)}$.

6.3 Phase transitions

The effective action obtained by the Legendre transform (6.25) may be seen as thermodynamic potential. By analogy with a magnetic system, one may think of the external source as a magnetic field, $J \sim B$, and of the variable Φ as a magnetization $\Phi \sim M$. The Legendre transform then defines the Gibbs free energy as a functional of the magnetization, so that it satisfies (6.28). The most stable state is the minimum of $\Gamma[\Phi]$. If $J = 0$ the Gibbs free energy reaches an extremum

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(\mathbf{x})} = 0 \tag{6.32}$$

at the corresponding value of Φ. If this equation has a non-trivial solution this signifies that a symmetry of the underlying theory is spontaneously broken. It is often very useful to write for $\Gamma[\Phi]$ some approximation

$$\Gamma[\Phi] = \int d^3x \left[\tfrac{1}{2}\alpha(T)(\nabla\Phi)^2 + \tfrac{1}{2}\mu(T)\Phi^2 + \tfrac{1}{4}\lambda(T)\Phi^4\right] , \tag{6.33}$$

where the parameters $\alpha(T), \mu(T), \lambda(T)$ which may depend on the temperature are treated as phenomenological input. The particular expansion (6.33) is called the "Ginzburg-Landau (GL)" model which is very successful in describing second-order phase transitions.

In the following we will assume that the states are translationally invariant. Then for each possible state, the solution will be a constant independent of the coordinate $\mathbf{x}$. Moreover we may assume that Γ, since it is a thermodynamic potential, is extensive, that is, it is proportional to the volume, and we may write

$$\frac{1}{V}\Gamma[\Phi] = V_{\text{eff}}(\Phi) . \tag{6.34}$$

This is the effective potential. The extremum condition then reduces to an ordinary differential equation

$$\frac{\partial V_{\text{eff}}(\Phi)}{\partial\Phi} = 0 \tag{6.35}$$

Each solution is a translationally-invariant state with $J = 0$. Equation (6.25) then implies that V_{eff} is equal to the thermodynamic potential of that state: $V_{\text{eff}} = \Omega$. Non-trivial solutions of equation (6.35) signify the occurrence of phase transition. For example, in

the case of a ferromagnet this equation has a non-trivial solution for temperatures below the Curie temperature.

For translationally invariant states, the GL-model reduces to the phenomenological Landau theory of second-order phase transitions. The basic assumption is that close to the critical temperature it is sufficient to expand the effective potential to fourth order

$$V_{\text{eff}}(\Phi) = V_{\text{eff}}(0) + \tfrac{1}{2}\mu(T)\Phi^2 + \tfrac{1}{4}\lambda(T)\Phi^4 \ . \qquad (6.36)$$

This effective potential has a minimum determined by

$$\mu(T)\Phi + \lambda(T)\Phi^3 = 0 \ . \qquad (6.37)$$

To have a phase transition, the coefficient $\mu(T)$ must flip sign at the critical temperature T_c: $\mu(T_c) = 0$, and $\lambda(T_c) > 0$. In the phase where $\mu(T)$ is positive, the effective potential is strictly convex while in the other regime it has the shape of a double-well potential with degenerate minima at $\Phi_0 = \pm\sqrt{-\mu/\lambda}$; in the absence of an external source both have the same free energy. Introduction of a small external field lifts the degeneracy and the so-called Landau function

$$\mathcal{L}(\Phi, J) \equiv \Gamma[\Phi] - J \cdot \Phi \qquad (6.38)$$

has a unique minimum. In that case the symmetry of the action $\Phi \to -\Phi$ is broken, and the system is said to be in a broken (asymmetric) phase.

Comparing the Landau-model (6.36) with the general expansion of the effective action we see that the Landau coefficients may be identified with the vertex functions integrated over all coordinates and divided by a volume factor:

$$V\tilde{\Gamma}^{(N)}(0) \equiv \int d^3x_1 \ldots d^3x_N \ \Gamma^{(N)}(\mathbf{x}_1, \ldots, \mathbf{x}_N) \ . \qquad (6.39)$$

That is, $\tilde{\Gamma}^{(2)}(0) = \mu(T)$ and $\tilde{\Gamma}^{(4)}(0) = 6\lambda(T)$. Hence by calculating these vertex functions one can in principle determine the critical temperature. For example, if one calculates the one-loop expression in thermal field theory, one finds the thermal mass $\sim T^2$. Hence, if one starts out with a negative bare mass, at a certain temperature a symmetry restoring phase transition must take place as first noticed by Dolan and Jackiw (1973). This observation has important implications for cosmology because it implies that any symmetry that is broken at present, must eventually be restored if one goes back to early enough times.

6.4 $\lambda\phi^4$-model

To proceed with the explicit calculation of the constrained effective action (CEA) we consider the $\lambda\phi^4$ model at temperature T. On the Euclidean contour we have

$$S^E[\phi] = \int_0^{\beta\hbar} d\tau \int d^3x \ [\tfrac{1}{2}(\partial_\tau\phi)^2 + \tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{4!}\lambda\phi^4] \ . \qquad (6.40)$$

The coupling constant λ has dimensions of inverse length times inverse energy. Therefore, the dimensionless expansion coefficient that may be used in the perturbative expansion is $g^2 \sim \lambda\hbar$ which is assumed to be small $g \ll 1$. Expanding into Matsubara modes, we may write the action as

$$
\begin{aligned}
S^E &= \int d^3x \left[\tfrac{1}{2} \sum_n \phi_n(\mathbf{x})(\omega_n^2 - \nabla^2 + m^2)\phi_{-n}(\mathbf{x}) \right. \\
&+ \left. \frac{\lambda T}{24} \sum_n \delta(n_1 + n_2 + n_3 + n_4) \prod_{j=1}^4 \phi_{n_j}(\mathbf{x}) \right] ,
\end{aligned} \tag{6.41}
$$

where the energy is conserved by the Kronecker delta; $\delta(n) = \beta\delta_{n,0}$. The object is to integrate out the "heavy" modes with $n \neq 0$ to obtain the CEA:

$$
\begin{aligned}
e^{-U[\Phi]} &= \mathcal{N} \int \mathcal{D}\phi \, \delta(\Phi - \mathcal{P}\phi) \, e^{-S^E[\phi]} \\
&= \mathcal{N} \int \mathcal{D}\phi_{n\neq 0} \, e^{-S^E[\Phi, \phi_{n\neq 0}]} .
\end{aligned} \tag{6.42}
$$

The action may be decomposed into three contributions

$$
S^E[\Phi, \phi_{n\neq 0}] = \int d^3x [\mathcal{H} + \mathcal{L}_0' + \mathcal{L}_I'] , \tag{6.43}
$$

where

$$
\mathcal{H}(\Phi) = \tfrac{1}{2}\Phi(\mathbf{x})(-\nabla^2 + m^2)\Phi(\mathbf{x}) + \tfrac{1}{4!}\lambda\Phi^4(\mathbf{x}) \tag{6.44}
$$

is the Hamiltonian density of the static field. The non-static modes appear in the two other contributions with a free Lagrangian of the massive modes

$$
\mathcal{L}_0' = \frac{1}{2}T \sum_n{}' \phi_n(\mathbf{x})(\omega_n^2 - \nabla^2 + m^2)\phi_{-n}(\mathbf{x}) \tag{6.45}
$$

and an interaction part

$$
\mathcal{L}_I' = \frac{\lambda}{24}T \sum_{\{n\}}{}' \delta(n_1 + n_2 + n_3 + n_4) \prod_{j=1}^4 \phi_{n_j}(\mathbf{x}) . \tag{6.46}
$$

In the sum $\sum_{\{n\}}'$ the term with all n's equal to zero is to be omitted. The last two contributions look very much like a zero-temperature action in terms of an infinite number of massive fields $\phi_n(\mathbf{x})$, with a temperature dependent mass $\omega_n = 2\pi nT$, which goes to infinity as $T \to \infty$. For this reason they are often called "heavy" modes in contrast to the "soft" modes $\Phi(\mathbf{x})$. We will use the following nomenclature (see also Kajantie, Laine, and Rummukainen and Shaposhnikov, 1996):

- heavy/hard modes: fields with mass and/or momentum $\sim T$

- soft modes: fields with mass and/or momentum $\sim gT$

- light modes: fields with mass and/or momentum $g^2 T$ or less with coupling constant $g^2 \sim \lambda\hbar$.

The CEA for the static modes appears as

$$e^{-U[\Phi]} = e^{-H} \int \mathcal{D}\phi_{n\neq0} \, \exp - \int d^3x \, (\mathcal{L}'_0 + \mathcal{L}'_I) \,, \tag{6.47}$$

with the Hamiltonian $H = \int d^3\mathcal{H}$. The crucial step is now that the heavy modes interact weakly with each other and with the light modes, when $g^2 \ll 1$. Therefore they can be integrated out with the use of perturbation theory, so that $U[\Phi]$ containing only the zero-mode can be constructed. From the Lagrangian we immediately read off the propagator for the non-static modes

$$\tilde{\Delta}(\omega_n, \mathbf{k}) = \frac{1}{\omega_n^2 + \omega_k^2} \tag{6.48}$$

which is just the Matsubara propagator.

To facilitate the perturbative calculation we introduce the following notation: the field Φ, $n = 0$ is indicated by a dashed line, and the sum over fields ϕ_n, $n \neq$ by a solid line. The interaction term in (6.46) are then represented diagrammatically by three interaction vertices

$$\mathcal{L}'_I = \frac{1}{4} \quad \times \quad + \frac{1}{3!} \quad \times \quad + \frac{1}{4!} \quad \times \tag{6.49}$$

The CEA of the effective theory may than be calculated from

$$U[\Phi] = H[\Phi] + \tfrac{1}{2}\text{Tr}' \log \tilde{\Delta}^{-1} + \langle e^{- \int d^3x \, \mathcal{L}'_I} - 1 \rangle_{\text{con}} \,, \tag{6.50}$$

where the second term at the right-hand side is the partition function of the non-interacting non-static mode:

$$Z'_0 = \int \mathcal{D}\phi_{n\neq0} \exp - \int d^3x \mathcal{L}'_0 = \exp \tfrac{1}{2}\text{Tr} \log \tilde{\Delta} \,. \tag{6.51}$$

To compute Γ we need to consider all graphs with all external lines corresponding to the $n = 0$ mode and all the internal lines corresponding to the $n \neq 0$ propagator. No IR-divergences arise because no static mode is allowed on the internal lines. Therefore one may expand in powers of m^2/T^2.

The first two terms at the right-hand side of (6.50) are the well known contributions of the tree-graphs and the one-loop graph

$$\bigcirc = T \sum_{n\neq0} \int \frac{d^3k}{(2\pi)^3} \log(\omega_n^2 + \mathbf{k}^2 + m^2) \,. \tag{6.52}$$

The calculation of the last term in (6.50) proceeds in the usual manner, except that only 1PI diagrams are present, because the field $\phi_{n\neq0}$ does not contain the $n = 0$ mode. Hence we may write the CEA as a sum of two terms

$$U[\Phi] = H(\Phi) + \Gamma'[\Phi] \,. \tag{6.53}$$

The second term is the effective action as usually defined, but calculated for non-static modes. The vertex functions as defined in (6.26) determine the corrections to the mass and the coupling constant of the Hamiltonian of the static theory.

6.5 Problems with DR

The above results for the CEA of the $\lambda\phi^4$-model are formally exact. However, it quickly turns out that in practical calculations three problems arise (Braaten and Nieto, 1996). To begin with, the UV divergences associated with the original 4D theory can readily be removed by standard renormalization prescriptions. However, the 3D effective theory also has UV-divergences which must be regularized by introducing an UV-cutoff Λ. The UV-divergences of the effective theory includes power UV-divergences of the form $\Lambda^p, p = 1, 2, \dots$. Since the full theory is renormalizable, power divergences of the form Λ^p from loop integrals must be completely canceled by terms proportional to Λ^p in the vertex functions of the static part. Moreover, the power divergences depend on the regularization procedure and are simply regularization artefacts. For this reason it is convenient to use dimensional regularization in which momentum integrals are analytically continued to $d - 3\epsilon$ spatial dimensions

$$\int \frac{d^3k}{(2\pi)^3} \to \nu^{2\epsilon} \int \frac{d^dk}{(2\pi)^d} \tag{6.54}$$

with ν an arbitrary momentum scale. Power divergences are set equal to zero in this method, because integrals without momentum scale are zero by definition in dimensional regularization.

In contrast, logarithmic UV divergences $\log \Lambda/m$ from a loop integral must match onto a $\log T/\Lambda$ term in one of the parameters of $\Gamma[\Phi]$ in order for the Λ-dependence to cancel. Thus, the logarithmic divergences have real physical significance. Although these logarithms cancel also, the problem is that in the effective theory they are hidden in complicated integrals. To have a finite theory the Λ-dependent counter terms must be added explicitly.

Let us take a detailed look at the simple example of the one-loop self energy. First, the static contribution in the high-T limit is (see section 3.4):

$$\Pi_{n=0} = \tfrac{1}{2}\lambda T \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_k^2} = \frac{1}{4\pi^2}\lambda\Lambda T - \frac{1}{8\pi}\lambda m T \ . \tag{6.55}$$

The linear divergence indicates that the integration in the static mode is not dominated by the low-momenta as one perhaps would have expected.

The contribution of non-static modes is

$$\Pi_{n\neq 0} = \tfrac{1}{2}\lambda \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k}[1 + 2\bar{n}(\omega_k)] \ , \tag{6.56}$$

where $\bar{n}(\omega_k)$ is the Bose-Einstein distribution function with the zero mode subtracted

$$\bar{n}(\omega_k) = \frac{1}{e^{\beta\hbar\omega_k} - 1} - \frac{1}{\beta\hbar\omega_k} \simeq \beta\hbar\omega_k + \dots \tag{6.57}$$

As already explained in the introduction, the static terms are classical. Indeed, in the classical limit we have $n(\omega_k) \to T/\hbar\omega_k$. The one-loop contribution is multiplied by $\hbar$

which drops out. This is not surprising as the high-temperature limit $\sim$ classical limit, because in both cases the dimensionless small parameter is $\beta\hbar\omega_k \to 0$. For mass $m = 0$ the finite-T part is easily evaluated; see section 3.4.

$$\Pi_{n\neq0} = \frac{\lambda T^2}{24\hbar} + \tfrac{1}{2}\lambda\hbar \int \frac{d^3k}{(2\pi)^3} \left(\frac{1}{2|\mathbf{k}|} - \frac{T}{\hbar|\mathbf{k}|^2} \right) . \tag{6.58}$$

The first term is the thermal mass, and we see explicitly that the thermal mass comes from the non-static modes. The quadratic- and linear divergences are zero in dimensional regularization. However we will use a finite cutoff and then get:

$$\Pi_{n\neq0} = \frac{\lambda T^2}{24\hbar} - \frac{1}{4\pi^2}\lambda\Lambda T , \tag{6.59}$$

which, of course, has the same divergence as the static term, as it should, because the total result is the thermal mass.

We conclude that at one loop the static theory acquires a thermal mass of order gT, so that the static mode is no longer massless. By the restriction to non-static modes, $n \to \hat{n} = n - T/\hbar\omega_k$, the thermal mass is accompanied by a linear divergence $\sim \Lambda T$ that acts as a counter term for the same linear divergence in the static theory. Effectively the result is that the classical divergence is replaced by the thermal mass. In fact by choosing the cutoff equal to $\Lambda = T\hbar$ the non-static contribution disappears altogether and the mass term in classical theory is replaced by the thermal mass. Hence, what we learn from this discussion is that the effective classical theory must incorporated the thermal mass.

6.6 High-temperature behavior

The one-loop thermal mass M_T is proportional to T^2. The origin of this behavior is rather obvious. Classically, the loop integral would be quadratically divergent, but the Bose-Einstein factor cuts the integral off at $|\mathbf{k}| \sim T$, so that the quadratic divergence is translated into a T^2 behavior. That is, the leading contribution in T comes from loop momenta of order T. In the nomenclature of Braaten en Pisarski this is called a "hard thermal loop" (HTL) contribution.

In $\lambda\phi^4$-theory, the thermal mass is the only HTL contribution. Basically the reason is that a 3D $\lambda\phi^4$ theory is super-renormalizable, that is, only the one- and two-loop integrals are UV-divergent. Let us, for example, calculate the four-point vertex function at zero momentum

$$\Gamma^{(4)} = \tfrac{1}{2}\lambda^2 T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{(\omega_n^2 + \omega_k^2)^2} . \tag{6.60}$$

Again we consider the zero-mode and non-zero modes separately. The zero-mode contributes in the high-temperature limit

$$\Gamma^{(4)}_{n=0} = \frac{1}{16\pi}\lambda^2\frac{T}{m} , \tag{6.61}$$

and the sum over the non-zero modes

$$\Gamma_{n \neq 0}^{(4)} = \frac{1}{16\pi^2} \lambda^2 \hbar \log\left(\frac{T}{m\hbar}\right) \; . \tag{6.62}$$

Both expressions are finite, but they blow up for $m \to 0$. Also in this limit two-loop diagrams dominate over one-loop diagrams. Consider for instance a self energy insertion on a propagator line in the four-point vertex. It it can be estimated to be of order

$$\Gamma^{(4),2L} \sim \Gamma^{(4),1L} \times \frac{\lambda T^2}{m^2 \hbar}. \tag{6.63}$$

We see that when $\lambda T^2 / m^2 \hbar >> 1$, the two-loop contribution dominates over the one-loop one. Hence, the perturbation expansion is not valid. This is not unexpected since we have encountered the same IR-problem when dealing with the thermal mass. The solution to this problem has been discussed in section 3.4: daisy diagrams have to resummed. This amounts to the replacement of the vacuum mass by the thermal mass $m \to M_T$.

In the resummed theory there is, besides λ, another expansion parameter, namely:

$$\lambda T / M_T \sim \lambda^{\frac{1}{2}} \sim g. \tag{6.64}$$

For instance, we see this expansion parameter appear in (6.60) compared with the tree-level contribution λ. More generally, the occurrence of this expansion parameter can be seen as follows. We consider a diagram and add a loop to it, while we keep the number of external lines fixed. This brings in an extra interaction λ, an extra integration $T \int d^3k$, and two extra propagators $(\mathbf{k}^2 + M_T^2)^{-2}$. Provided the integrations give a finite result, the typical scale of the momentum is given by the mass M_T. The total result is the expansion parameter $\lambda T / m_T$.

An important implication is that in the resummed theory orders m/T do not vanish, but are replaced by $M_T / T \to \sim g$. This is of direct relevance for the validity of dimensional reduction, because it means that the zero modes do not completely decouple from the heavy modes $\sim 2\pi T$, only to a certain order in the coupling g. Still for $g \ll 1$ the scheme can be put to work.

This analysis may be extended to higher-order vertex functions. We consider the one-loop contributions, which are of order g^N. The static propagators have the IR-cut off $M_T \sim gT$, whereas the scale for the non-static propagators is set by $2\pi T$. Let $V = \#$ of vertices, $N = \#$ of external legs, $I = \#$ of internal lines, and $L = \#$ of loops in a given diagram. The superficial degree of divergence of the diagram is then $D = 3L - 2I$. Each loop summation over the Matsubara frequencies yield a factor T. If power UV-divergences are subtracted as part of the regularization scheme, the finite part must have a coefficient that is proportional to M_T^D in the static case and T^D in the latter. We are thus led to a characterization of the 1PI-graphs by their order in g and T:

$$\Gamma_{n=0}^{(N)} \sim g^N T M_T^D = \lambda^N \frac{T}{M_T^{N-3}} \quad N \geq 3 \; , \tag{6.65}$$

$$\Gamma_{n \neq 0}^{(N)} \sim g^N T^{D+1} = \lambda^N \frac{1}{T^{N-4}} \quad N \geq 3. \tag{6.66}$$

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where we used the well known relationships $L = I - V + 1$, $N = 4V - 2I$. Since $M_T \sim gT$, the effect of the replacement $m \to M_T$ is that

$$\Gamma_{n=0}^{(N)} \sim g^3 \frac{1}{T^{N-4}} \quad N \geq 3 . \tag{6.67}$$

The static higher orders are suppressed by the temperature, but not by the coupling constant, in contrast to the non-zero modes. There is no need to replace the vacuum mass in the propagator for the non-zero modes. This will only give subleading corrections, since $g^2 T^2 \ll \omega_{n \neq 0}^2 \sim T^2$. In a systematic expansion the resummation of thermal corrections to the mass is only necessary in the zero-mode propagator, whereas the hard-mode contributions are perturbatively calculable without resummation;

6.7 Matching

Let us summarize what has been achieved so far. By integrating out the non-static modes, we have derived an effective theory that in principle allows us to calculate the static correlation functions of the full theory. The basic identity is:

$$\int \mathcal{D}\phi \, e^{-S^E[\phi]} \mathcal{P}\phi(x_1) \dots \mathcal{P}\phi(x_N) = \mathcal{N} \int \mathcal{D}\Phi \, e^{-U[\Phi]} \Phi(\mathbf{x}_1) \dots \Phi(\mathbf{x}_N) , \tag{6.68}$$

which is an exact consequence of dimensional reduction. In principle the CEA $U[\Phi]$ still contains all necessary information about the system sufficient to calculate static correlation functions to any desired degree of accuracy. However this has a price, namely, $U[\Phi]$ is defined by an infinite sum of vertex functions, which are moreover non-local in general. Therefore, the whole scheme of DR is useful only if there is some approximation scheme that allows us to simplify and truncate the infinite sum. The approximation we could consider is a local effective theory

$$U[\Phi] = \int d^3x \, \mathcal{L}_{\text{eff}} , \tag{6.69}$$

where

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{2}\alpha(T)(\nabla\Phi)^2 + \tfrac{1}{2}\mu(T)\Phi^2 + \tfrac{1}{4}\lambda(T)\Phi^4 + \dots , \tag{6.70}$$

includes all local terms that can be built from the static fields consistent with the symmetries of the system. The parameters $\alpha(T), \mu(T), \lambda(T), \dots$ and all other parameters that one could introduce, depend on the temperature, the coupling constants of the underlying theory, and some UV-cut off Λ.

We have already seen that higher-order loop corrections are small. Hence, as the simplest approximation, the parameters in the effective Lagrangian are determined by the one-loop corrections to the vertex functions, in the small-momentum limit:$\alpha(T) = 1$,

$$\mu(T) = \tilde{\Pi}(0, \mathbf{k})\big|_{\mathbf{k}=0} + \mathcal{O}(\lambda^2) = \frac{\lambda T^2}{24} , \tag{6.71}$$

$$\lambda(T) = \tilde{\Gamma}^{(4)}(0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\big|_{\mathbf{k}=0} + \mathcal{O}(\lambda^2) = \lambda . \tag{6.72}$$

Figure 6.2: Static an non-static tadpole diagrams

Now it was observed by Landsman (1989) that an effective theory constructed in this way, does not completely reproduce the underlying theory, if one confines oneself to a renormalizable effective theory. Moreover, the results depend on the renormalization scheme. However, this does not imply that DR fails, but simply means that non-renormalizable operators must be included in $\mathcal{L}_{\text{eff}}$ in order to extend the accuracy to higher orders of λ. The resulting theory is non-renormalizable and has in general infinitely many parameters. These parameters should be chosen such thar the effective theory reproduces the long-distance behavior of the original theory. This procedure is known as *matching*, that is, compute the corresponding quantities in the effective theory, and demand that they match.

Let us now discuss how matching works in practical applications. We shall outline how one determines $\mu(T)$ and $\lambda(T)$ at the two-loop level (Andersen 1997). We must start by determining what "matching" conditions we are going to use. In principle, there should be as many matching conditions as there are unknown constants. It seem natural to restrict the discussion to the effective parameters to $\alpha(T)$, $\mu(T)$, $\lambda(T)$ (then the 3D Lagrangian is super-renormalizable) and to match the simplest Green functions that appear in the 3D theory:

$$\tilde{\Pi}(0, \mathbf{k})\Big|_{\mathbf{k}=0} = \tilde{\Pi}_{\text{eff}}(\mathbf{k})\Big|_{\mathbf{k}=0} + \mathcal{O}(\lambda^3) , \tag{6.73}$$

$$\partial_{|\mathbf{k}|^2}\tilde{\Pi}(0, \mathbf{k})\Big|_{\mathbf{k}=0} = \partial_{|\mathbf{k}|^2}\tilde{\Pi}_{\text{eff}}(\mathbf{k})\Big|_{\mathbf{k}=0} + \mathcal{O}(\lambda^3) , \tag{6.74}$$

$$\tilde{\Gamma}^{(4)}(0, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\Big|_{\mathbf{k}=0} = \tilde{\Gamma}^{(4)}_{\text{eff}}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)\Big|_{\mathbf{k}=0} + \mathcal{O}(\lambda^3) . \tag{6.75}$$

This yields three equations for three coefficients $\alpha(T)$, $\mu(T)$, $\lambda(T)$.

At the one-loop level in the full theory, there is only one contributing diagram, namely the tadpole, which may be split into a static and non-static part

$$\tilde{\Pi}(\mathbf{k}) = \mathbf{k}^2 + \tilde{\Pi}_{n=0}(\mathbf{k}) + \tilde{\Pi}_{n\neq 0}(\mathbf{k}) . \tag{6.76}$$

We have already calculated the last term in the high-T limit.

$$\tilde{\Pi}_{n\neq 0}(\mathbf{k}) = \frac{\lambda}{24}T^2 - \frac{\lambda}{(2\pi)^2}\Lambda T . \tag{6.77}$$

On the other hand, in the effective theory the corresponding expression is

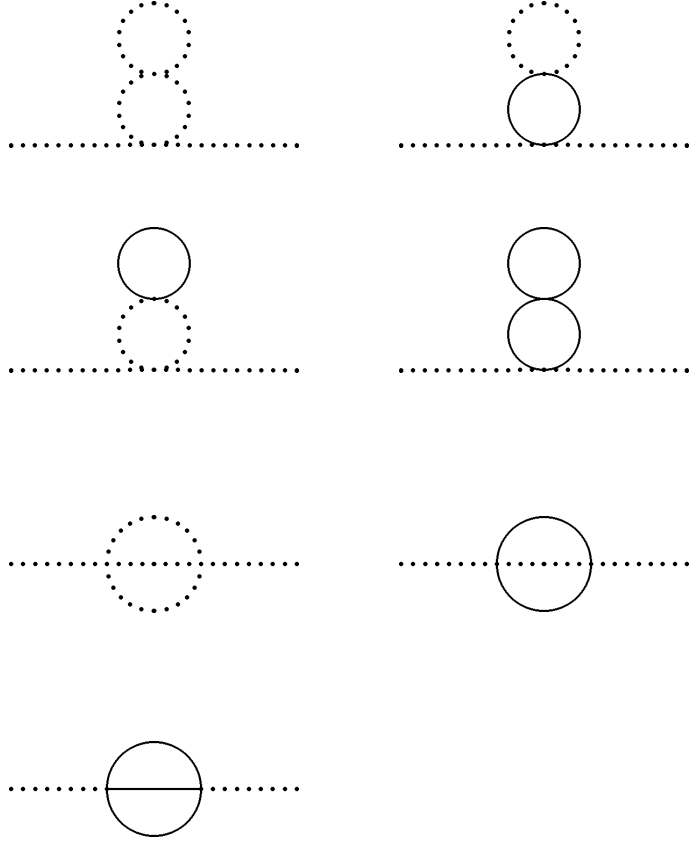$$\Pi_{\text{eff}}(\mathbf{k})|_{\mathbf{k}=0} = \quad \text{} \tag{6.78}$$

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Figure 6.3: The two-loop graphs for the two-point function

The demand that these two expressions are the same, determines $\alpha(T)$, $\mu(T)$, $\lambda(T)$ to one-loop order:

$$\mu(T) = \frac{\lambda T^2}{24} + \mathcal{O}(\lambda^2) \,, \tag{6.79}$$

$$\lambda(t) = \lambda + \mathcal{O}(\lambda^2) \,. \tag{6.80}$$

Hence, to one-loop order the matching procedure, coincides with the DR prescription (6.72) for the coupling constant as found in the original papers on DR.

At 2-loop level things become more interesting. One then encounters the essential non-locality of the theory. As we will see, a non-local vertex has to be incorporated in the theory. First we write down all that diagrams that contribute to the self energy in the full=DR theory; see Fig 6.3. The static modes are indicated by a dotted line and non-static modes by a solid line. This has to be matched to the two-loop graphs of the effective theory.

The diagrams with only light modes, cancel against the corresponding graphs in the effective theory. To second order one needs to include the loop-corrections to the four-
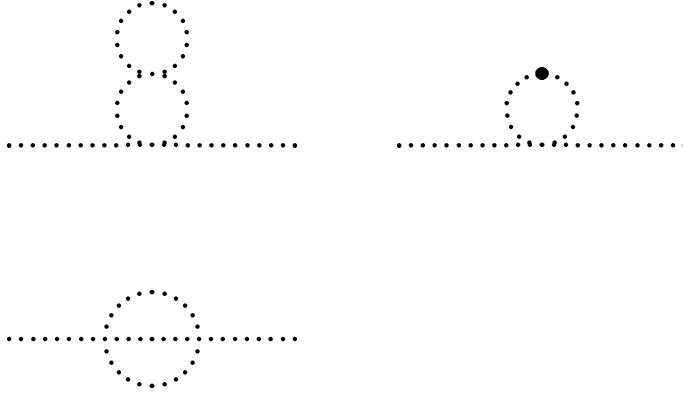
Figure 6.4: The two-loop graphs for the two-point function in the effective theory

point functions.

$$\tilde{\Gamma}^{(4)}(0, \mathbf{k} = 0) = \quad\text{⟨diagram⟩} \quad + \quad\text{⟨diagram⟩} \quad , \tag{6.81}$$

$$\tilde{\Gamma}^{(4)}_{\text{eff}}(\mathbf{k} = 0) = \quad\text{⟨diagram⟩} \quad . \tag{6.82}$$

Hence, to this order the matching of the coupling constant reduces to

$$\lambda(T) = \quad\text{⟨diagram⟩} \quad = \quad\text{⟨diagram⟩} \quad + \quad\text{⟨diagram⟩} \quad . \tag{6.83}$$

This effective vertex has been introduced by Jakovac (1997).

Now the matching equations have to be solved for $\mu(T)$. The most interesting parts of the DR vertex function are the setting sun diagrams. However they turn out to be zero in dimensional regularization. This means that we can ignore the non-local part. The final result is that we have the following matching relation for the mass (see also Braaten and Nieto, 1996):

$$\begin{aligned}
\mu(T) &= \tfrac{1}{2}\lambda T \nu^{2\epsilon} \sum_{n \neq 0} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\omega_n^2 + \omega_k^2} \\
&\quad - \tfrac{1}{4}\lambda^2 \nu^{4\epsilon} \sum_{n \neq 0} \sum_{l \neq 0} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\omega_n^2 + \omega_k^2} \frac{1}{(\omega_l^2 + \omega_q^2)^2} .
\end{aligned} \tag{6.84}$$

Apart from zero-temperature renormalizations one finds:

$$\mu(T) = \frac{\lambda T^2}{24} \left\{ 1 + \frac{\lambda}{16\pi^2} \left[\log \frac{\Lambda}{4\pi T} - \gamma_E + 2 + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right] \right\} . \tag{6.85}$$

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The logarithmic Λ-dependence acts as a counter term and neutralizes logarithmic UV-divergences from loop integrals in the effective theory, so that the effective theory is cut-off independent to this order. We note that the IR-divergences that arise in the underlying theory precisely match those in the effective theory. This is a general feature of the matching procedure.

Appendix
Ideal Bose and Fermi gas

We review some properties of non-interacting Bose and Fermi gases consisting of particles with energy ϵ_k. The general expression for the thermodynamic potential is

$$\Omega = \eta \int \frac{d^3 k}{(2\pi)^3} \log \left[1 - \eta e^{-\beta(\epsilon_k - \mu)} \right] , \tag{6.86}$$

where μ is the chemical potential and $\eta = 1$ for bosons and $\eta = -1$ for fermions. After an integration by parts we get

$$\Omega = -\frac{1}{6\pi^2} \int_{\omega_0}^{\infty} d\omega \, \mathbf{k}^3 n(\omega) \tag{6.87}$$

where $\omega_0 = \beta(\epsilon_0 - \mu)$ is the value of the exponent for $|\mathbf{k}| = 0$, and

$$n(\omega) = \frac{1}{e^{\omega} - \eta} \tag{6.88}$$

is the Fermi or Bose equilibrium distribution function.

For $m = 0$ the thermodynamic potential (6.87) can be expressed in terms of the standard Bose or Fermi intergrals generally defined by

$$F_n(y) = \frac{1}{\Gamma(n)} \int_0^{\infty} \frac{x^{n-1}}{e^{x-y} - \eta} . \tag{6.89}$$

Here $\Gamma(n)$ is the gamma function with the properties $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. The integrals satisfy the recursion relation

$$\frac{dF_n(y)}{dy} = F_{n-1}(y) . \tag{6.90}$$

For the case $\beta\mu = y = 0$ they can be evaluated analytically with the help of the formulae

$$\int_0^{\infty} \frac{x^{n-1}}{e^x - 1} = \Gamma(n)\zeta(n), \qquad n > 1 \tag{6.91}$$

$$\int_0^{\infty} \frac{x^{n-1}}{e^x + 1} = (1 - 2^{1-n})\Gamma(n)\zeta(n), \qquad n > 0 \tag{6.92}$$

Values of the zeta function that often occur are: $\zeta(2) = \pi^2/6, \zeta(3) = 1.202, \zeta(4) = \pi^4/90$.

The thermodynamic potential for massless bosons with one degree of freedom and $\mu = 0$ is

$$\Omega = -\frac{T^3}{6\pi^2\hbar^3} \int_0^{\infty} \frac{x^3}{e^x - 1} = \frac{\pi^2 T^3}{90\hbar^3} . \tag{6.93}$$

From this equation we get the pressure as $P = -T\Omega$ and the energy density $E = 3P$. The particle density is

$$N = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{e^{\beta \hbar |\mathbf{k}|} - 1} = \frac{T^3}{\pi^2 \hbar^3} \zeta(3) , \qquad (6.94)$$

which gives for the entropy density $S = 4E/3T \simeq 3.6N$.

Let us now turn to massless fermions. Taking particles an anti-particles into account one has

$$\Omega = -\frac{T^3}{\pi^2 \hbar^3} [F_4(y) - F_4(-y)] . \qquad (6.95)$$

Remarkably, this difference, and similar combinations

$$S_n = F_n(y) + (-1)^n F_n(-y) , \qquad (6.96)$$

can all be computed exactly. For $n = 1$ we have

$$F_1(y) = \log(1 + e^y) = y + \log(1 + e^{-y}) , \qquad (6.97)$$

and thus $S_1(y) = y$. If one now uses the recursion relation (6.90), one may calculate subsequent orders by integration:

$$2! S_2 = y^2 + \tfrac{1}{3}\pi^2 , \qquad (6.98)$$
$$3! S_3 = y^3 + \pi^2 y , \qquad (6.99)$$
$$4! S_4 = y^4 + 2\pi^2 y^2 + \frac{7\pi^4}{15} . \qquad (6.100)$$
$$\qquad (6.101)$$

The last result leads to:

$$\Omega = -\frac{T^3}{6\pi^2 \hbar^3} \left[\frac{7\pi^4}{60} + \frac{\pi^2 \mu^2}{2T^2} + \frac{\mu^4}{4T^4} \right] . \qquad (6.102)$$

The particle density may be calculated by differentiation with respect to y.

Appendix
Gaussian Integrals

Gaussian integrals have some important applications. One of those is that they form the basis of perturbation theory in quantum field theory. In this appendix some algebraic identities about Gaussian integrals are briefly described for a finite number of integration variables.

Throughout the appendix the *Einstein summation convention* is used, that is, summation is understood to be carried out over repeated index symbols.

A general Gaussian integral has the form:

$$Z(j_1, \ldots, j_N) = \int_{-\infty}^{\infty} \prod_{i=1}^{N} d\phi_i \exp\left(-\frac{1}{2}\phi_i A_{ij}\phi_j + j_i\phi_i\right) , \qquad (6.103)$$

in which A is a *symmetric positive definite* matrix, that is, its eigenvalues λ_i satisfy $\mathrm{Re}(\lambda_i) \geq 0$, $\lambda_i \neq 0$.

To calculate Z one first determines the maximum contribution of the integrand to the integral from the extremum of the exponent:

$$\frac{d}{d\phi_k}\left(\frac{1}{2}\phi_i A_{ij}\phi_j - j_i\phi_i\right) = 0 . \qquad (6.104)$$

Using the solution of this equation, we shift integration variables according to:

$$\phi_i = (A^{-1})_{ij}j_j + y_i . \qquad (6.105)$$

The integral (6.103) now becomes:

$$Z(j_1, \ldots, j_N) = Z[0]\exp\frac{1}{2}j_i(A^{-1})_{ij}j_j , \qquad (6.106)$$

where the Gaussian integrals

$$Z[0] = \int_{-\infty}^{\infty} \prod_{i=1}^{N} dy_i e^{-\frac{1}{2}y_i A_{ij}y_j} \qquad (6.107)$$

can be calculated by diagonalizing the matrix A. One finally obtains:

$$Z(j_1, \ldots, j_N) = (2\pi)^{\frac{N}{2}}(\det A)^{-\frac{1}{2}}\exp\frac{1}{2}j_i(A^{-1})_{ij}j_j \qquad (6.108)$$

The last expression can be used to calculate any Gaussian average:

$$\langle\phi_{k_1}\phi_{k_2}\ldots\phi_{k_l}\rangle \equiv \mathcal{N}\int_{-\infty}^{\infty}\left(\prod_{i=1}^{N} d\phi_i\right)\phi_{k_1}\phi_{k_2}\ldots\phi_{k_l}e^{-\frac{1}{2}\phi_i A_{ij}\phi_j} \qquad (6.109)$$

in which the normalization $\mathcal{N}$ is chosen in such a way that $< 1 >= 1$, i.e. $\mathcal{N}^{-1} = Z(0)$. Consider the general Gaussian integral (6.103). Repeated differentiation with respect to the sources leads to the identity:

$$\langle\phi_{k_1}\phi_{k_2}\ldots\phi_{k_l}\rangle = (2\pi)^{-\frac{N}{2}}(\det A)^{\frac{1}{2}}\frac{\partial}{\partial j_{k_1}}\frac{\partial}{\partial j_{k_2}}\cdots\frac{\partial}{\partial j_{k_l}}Z(j_1, \ldots, j_N)\bigg|_{\mathbf{j}=0} . \qquad (6.110)$$

Insertion of the explicit form (6.108) then gives

$$\langle\phi_{k_1}\ldots\phi_{k_l}\rangle = \frac{\partial}{\partial j_{k_1}}\cdots\frac{\partial}{\partial j_{k_l}}e^{\frac{1}{2}j_i(A^{-1})_{ij}j_j}\bigg|_{\mathbf{j}=0} \qquad (6.111)$$

$$= \frac{\partial}{\partial j_{k_1}}\cdots\frac{\partial}{\partial j_{k_l}}\frac{1}{N!}\frac{1}{2^N}\left(j_i(A^{-1})_{ij}j_j\right)^N , \qquad (6.112)$$

with $l = 2N$; the number of variables and indices must be even. In principle the differentiations yield $2N!$ different terms. However, since the matrix A is symmetric, 2^N of these terms are equal. Furthermore, we are differentiating a monomial of order N, which implies that there are $N!$ permutations that also yield the same terms. Hence the total number of different terms is $(2N!)/2^N N! = (2N-1)!!$. This is simply the number of all possible pairings of $l = 2N$ indices. Thus one finds:

$$\langle \phi_{k1} \ldots \phi_{kl} \rangle = \sum_{\substack{\text{pairings of} \\ (k_1,\ldots,k_l)}} A^{-1}_{k_{p_1} k_{p_2}} \ldots A^{-1}_{k_{p_{l-1}} k_{p_l}} . \qquad (6.113)$$

The rule is:

- consider all possible pairings of the indices $k_1, \ldots, k_l$ (l even),

- associate to each pair $k_p k_q$ a matrix element of the matrix A^{-1}

Identity (6.113) states that all moments of a Gaussian distribution can be expressed in terms of the second moment alone. Indeed, consider the second moment

$$\langle \phi_{k1} \phi_{k2} \rangle = \left(A^{-1} \right)_{k_1 k_2} . \qquad (6.114)$$

It is simply equal to the inverse of the matrix A. Therefore an alternative way of writing (6.113) is

$$\langle \phi_{k1} \ldots \phi_{kl} \rangle = \sum_{\substack{\text{pairings of} \\ (k_1,\ldots,k_l)}} \langle \phi_{k_{p_1}} \phi_{k_{p_2}} \rangle \ldots \langle \phi_{k_{p_{l-1}}} \phi_{k_{p_l}} \rangle \qquad (6.115)$$

In quantum field theory this result is known as *Wick's theorem* and the basis of perturbative calculations.

Appendix
Functional Differentiation

We first give a general definition: a functional F is a mapping from functions $\eta(x)$ on R^d onto the real or complex numbers (or in general on vectors). One writes $F[\eta]$ with square brackets to emphasize that the argument is a function rather than a number. A simple example of a functional is

$$F[\eta] = \int dx \, f(x) \eta(x) , \qquad (6.116)$$

with $f(x)$ some generalized function on R^d.

Next, we introduce the notion of the functional derivative of a functional. It is defined as a linear functional on a space of suitable test functions $\varphi(x)$ according to

$$\int dx \, \varphi(x) \frac{\delta F[\eta]}{\delta \eta(x)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \{ F[\eta + \epsilon \varphi] - F[\eta] \} . \qquad (6.117)$$

Applying this definition to the functional (6.116) , we get

$$\frac{\delta F[\eta]}{\delta \eta(x)} = f(x) \ . \tag{6.118}$$

In a similar manner we derive

$$\frac{\delta \eta(x)}{\delta \eta(y)} = \delta(x - y) \ . \tag{6.119}$$

The rules for functional differentiation are very much like the ones for ordinary derivatives. For, instance, under some mild continuity conditions we have

$$\frac{\delta}{\delta \eta(x)} \frac{\delta F[\eta]}{\delta \eta(y)} = \frac{\delta}{\delta \eta(y)} \frac{\delta F[\eta]}{\delta \eta(x)} = \frac{\delta^2 F[\eta]}{\delta \eta(x) \delta \eta(y)} \ , \tag{6.120}$$

or

$$\left[\frac{\delta}{\delta \eta(x)}, \frac{\delta}{\delta \eta(y)} \right] = 0 \ . \tag{6.121}$$

Other rules one may use are

$$\frac{\delta}{\delta \eta(x)} F^n[\eta] = n F^{n-1}[\eta] \frac{\delta F[\eta]}{\delta \eta(x)} \ , \tag{6.122}$$

$$\frac{\delta}{\delta \eta(x)} e^{F[\eta]} = e^{F[\eta]} \frac{\delta F[\eta]}{\delta \eta(x)} \ , \tag{6.123}$$

which imply

$$\left[\frac{\delta}{\delta \eta(x)}, \eta(y) \right] = \delta(x - y) \ . \tag{6.124}$$

These rules suffice for our purpose.

Under suitable differentiability conditions there also exists an analogue of the Taylor expansion for a functional. Indeed, by Taylor's formula we have

$$F[\eta + z\varphi] = F[\eta] + z \left. \frac{\partial F}{\partial z} \right|_{z=0} + \tfrac{1}{2} z^2 \left. \frac{\partial^2 F}{\partial z^2} \right|_{z=0} + \cdots \tag{6.125}$$

regarding F as an ordinary function of z. Using the definition (6.117) of the functional derivative, we may write

$$F[\eta + z\varphi] = F[\eta] + \sum_{N=1}^{\infty} \frac{z^N}{N!} \int dx^N \varphi(x_1) \ldots \varphi(x_N) \frac{\delta^N F[\eta]}{\delta \eta(x_1) \ldots \delta \eta(x_N)} \ . \tag{6.126}$$

Taking $\eta = 0, z = 1$, and changing φ to η, we obtain the so-called Volterra series:

$$F[\eta] = F[0] + \sum_{N=1}^{\infty} \frac{1}{N!} \int dx^N \eta(x_1) \ldots \eta(x_N) \left. \frac{\delta^N F[\eta]}{\delta \eta(x_1) \ldots \delta \eta(x_N)} \right|_{\eta=0} \ . \tag{6.127}$$
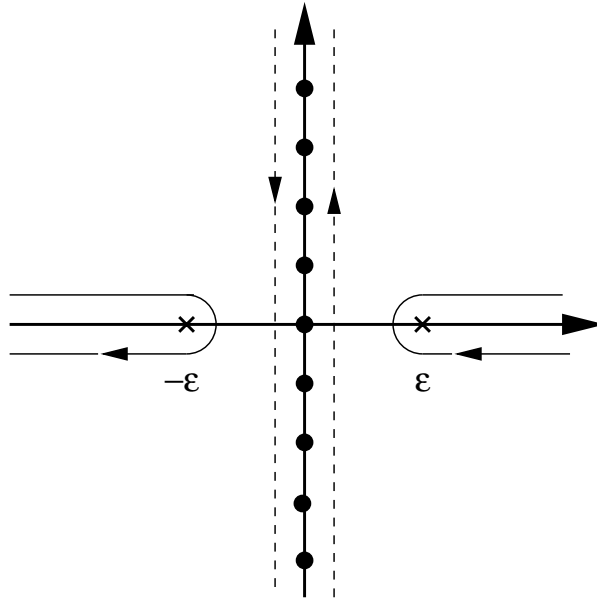
Figure 6.5: Integration contour used in calculation of Matsubara frequencies by analytic continuation

If this series converges, the functional $F[\eta]$ is completely specified by giving the infinite set of symmetric functions

$$G(x_1, \ldots, x_N) = \left. \frac{\delta^N F[\eta]}{\delta \eta(x_1) \ldots \delta \eta(x_N)} \right|_{\eta=0} \tag{6.128}$$

for which $F[\eta]$ is said to be the generating functional. This formalism is very useful as a starting point for perturbation theory.

Appendix
Analytic Continuation

The energy summations that occur in the Matsubara formalism may be performed by an analytic extension away from the discrete complex energies down the real axis. We discuss three approaches.

(1) The first one is standard and involves the replacement of energy sums by contour integrals. For a summation over bosonic energies $k_0 = i2n\pi T$ the rule is

$$T \sum_n f(k_0) = \int_{C^+} \frac{dz}{2\pi i} n(z)[f(z) + f(-z)] + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z) , \tag{6.129}$$

and for odd fermionic frequencies $k_0 = i(2n+1)\pi T - \mu$

$$T \sum_n f(k_0) = - \int_{C^+} \frac{dz}{2\pi i} [n_+(z)f(z) + n_-(z)f(-z)] + \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i} f(z) \ . \qquad (6.130)$$

These formulae are valid for any function $f(z)$ which is analytic in the neighborhood of the imaginary line $\mathrm{Re}\, z = \mu$, and which has the property that the product $f(z) \exp{-\beta|z|}$ vanishes sufficiently fast at infinity. The contour C_+ circumscribes *clockwise* all singularities of the functions $f(z)$ and $f(-z)$ in the right half plane, but none of the poles of the Fermi-Dirac and Bose-Einstein distribution functions

$$n_\pm(z) = \frac{1}{\exp \beta(z \mp \mu) + 1}, \quad n(z) = \frac{1}{\exp \beta z - 1} \ , \qquad (6.131)$$

whose poles occur at the Matsubara frequencies $z = i\omega_n$, each with residue T.

One may notice that no use is made of the convergence factor which plays a crucial role in non-relativistic many-body theory. This convergence factor enables one to close the vacuum contour in the left half plane. In a theory without anti-particles this amounts to a specific renormalization prescription to eliminate vacuum terms.

(2) The Matsubara sums can be performed with the help of the above formula. However, the choice of the pole generating functions is not unique and one may also do the sum by the so called "Coth method", in the case of bosons. Let $f(z)$ be a meromorphic function with no poles at any of the Matsubara frequencies, then

$$T \sum_n f(i\omega_n - \mu) = -\mathrm{Res}(f, z_i) \tfrac{1}{2} \coth \tfrac{1}{2}\beta(z_i - \mu) \ , \qquad (6.132)$$

where i runs over all poles z_i of the function $f(z)$ and $\mathrm{Res}(f, z_i)$ stands for the residue of $f(z)$ in the poles at $z = z_i$. This expression is valid provided $f(z)$ falls off rapidly enough at $|z| \to \infty$ to ensure that the contour integration of $f(z)$ along a complex contour enclosing all poles vanishes in this limit. The proof is a standard application of the residue theorem and uses the fact that the function $\tfrac{1}{2} \coth \tfrac{1}{2}\beta z$ has poles at $z = i\omega_n$ with unit residue.

There is an important corollary to this summation rule, which states that in a given diagram the number of distribution functions is directly related to the number of loops in the diagram. The proof is simple: in the imaginary-time formalism the number of loops equals the number of Matsubara frequency summations. In practical calculations it is convenient to perform a partional fractioning on the propagators, e.g.

$$\tilde{\Delta}(i\omega_n, \mathbf{k}) = \frac{1}{\omega_n^2 + \omega_k^2} = \sum_{s=\pm 1} \frac{1}{2\omega_k} \frac{s}{i\omega_n - s\omega_k} \ . \qquad (6.133)$$

Using the method of contour integration to perform the sums, each sum gives rise to one 'coth' function, either with positive or negative energy

$$\tfrac{1}{2} \coth \tfrac{1}{2}s\beta\omega = n(s\omega) + \tfrac{1}{2} = s[n(\omega) + \tfrac{1}{2}], \quad s = \pm 1 \ . \qquad (6.134)$$

Hence, the resulting expressions are of the form of spatial momentum integrals over Bose-Einstein or Fermi-Dirac distribution functions, where the number of distribution functions is equal to, or less, than the numbers of loops.

(3) There is also a method for direct analytic continuation of Matsubara diagrams without actually doing the frequency summations. The method was originally proposed by Balian and de Dominicis (1960), and independently by Dzyaloshinskii (1962) and by Baym and Sessler (1963). Recently the method has been rediscovered by Pisarski (1988) and in the thermal field literature it is now referred to as the "Saclay Method". The starting point is the contour propagator (1.63) in the time-representation, but Fourier-transformed in space

$$\Delta(\tau, \mathbf{k}) = \int \frac{dk_0}{2\pi} \rho_0(k) e^{-\tau k_0} [\theta(\tau) + \eta n(\omega)] \quad , \tag{6.135}$$

$$\tilde{\Delta}(k) = \int_0^\beta d\tau e^{k_0 \tau} \Delta(\tau, \mathbf{k}) \quad . \tag{6.136}$$

Reverting from frequency to time has the advantage that the frequency summations can now be trivially performed. The photon propagator has been calculated by this method in chapter 3.

A graphical representation is obtained by arranging the vertices in a given diagram such that their imaginary time coordinates decrease from right to left. The important point is that the intermediate times have a definite sign: on an internal line directed from right to left the sign is positive, and on a line with the opposite direction negative. In the former case the propagator contributes a factor $1 + \eta n$ and in the opposite case a factor ηn. The time integrations can then be performed. Finally the sum has to be taken over all time orderings.

Although straightforward in principle, the higher-order calculations become quite tedious and it is difficult to keep track of all contributions. The problem is that in the "Saclay method" one gets one distribution function for each propagator, whereas we know that the number of distribution functions after analytic continuation should be equal to or less than the number of loops. Hence, during the calculations a host of redundant terms must disappear through cancelations.

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